NEW ELEMENTARY ALGEBRA

EMBRACING

THE FIRST PRINCIPLES OF THE SCIENCE

BY

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PREFACE.

In the admirable series of mathematical text-books, still unequaled in many essential particulars, which the late Professor Charles Davies issued, the "Elementary Algebra" has an important place. The intent and scope of the work is set forth in the preface to the latest edition that the accomplished author prepared: —

"Algebra naturally follows arithmetic in a course of scientific studies. The language of figures, and the elementary combinations of numbers, are acquired at an early age. When the pupil passes to a new system, conducted by letters and signs, the change seems abrupt; and he often experiences much difficulty before perceiving that algebra is but arithmetic written in a different language.

"It is the design of this work to supply a connecting link between arithmetic and algebra; to indicate the unity of the methods; and to conduct the pupil from the arithmetical processes to the more abstract methods of analysis, by easy and simple gradations. The work is also introductory to the 'University Algebra,' and to the 'Algebra' of M. Bourdon, which latter is justly considered, both in this country and in Europe, as the best text-book on the subject which has yet appeared.

"In the 'Introduction,' or 'Mental Exercises,' the language of figures and letters are both employed. Each lesson is so arranged as to introduce a single principle not known before; and the whole is so combined as to prepare the pupil, by a thorough system of mental training, for those processes of reasoning which are peculiar to the algebraic analysis."

The definitions are precise; the fundamental principles and operations are lucidly explained; and the whole subject within the range of the treatise is simply and logically developed, and well illustrated by appropriate examples.

The present edition is the result of a careful reëxamination of the work, and in it are incorporated such emendations as the progress of educational science and practical experience in teaching have suggested.

JULY, 1891.

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SUGGESTIONS TO TEACHERS.

1. THE introduction is designed as a mental exercise. If thoroughly taught, it will train and prepare the mind of the pupil for those higher processes of reasoning which it is the peculiar province of the algebraic analysis to develop.

2. The statement of each question should be made, and every step in the solution gone through with, without the aid of a slate or blackboard, though perhaps in the beginning some aid may be necessary to those unaccustomed to such exercises.

3. Great care must be taken to have every principle on which the statement depends carefully analyzed; and equal care is necessary to have every step in the solution distinctly explained.

4. The reasoning process is the logical connection of distinct apprehensions, and the deduction of the consequences which follow from such a connection: hence the basis of all reasoning must lie in distinct elementary ideas.

5. Therefore, to teach one thing at a time, to teach that thing well, to explain its connections with other things and the consequences which follow from such connections, would seem to embrace the whole art of instruction.

ELEMENTARY ALGEBRA.

INTRODUCTION.

MENTAL EXERCISES.

LESSON I.

1. John and Charles have the same number of apples. Both together have 12. How many has each?

Let x denote the number which John has. Then, since they have an equal number, x will also denote the number which Charles has; and twice x, or $2x$, will denote the number which both have, which is 12. If twice x is equal to 12, x will be equal to 12 divided by 2, which is 6: therefore each has 6 apples. Hence we write, —

Let $\qquad x =$ number of apples John has.

Then $\qquad x + x = 2x = 12.$

Hence $\qquad x = \frac{12}{2} = 6.$

NOTE. — When x is written with the sign $+$ before it, it is read *plus x;* and the line above is read, *x plus x equals* 12.

When x is written by itself, it is read *one x*, and is the same as $1x$.

x or $1x$	means once	x,	or one x,
$2x$	" twice	x,	or two x,
$3x$	" three times	x,	or three x,
$4x$	" four times	x,	or four x, etc.

2. What is $x+x$ equal to?

3. What is $x+2x$ equal to?

4. What is $x+2x+x$ equal to?

5. What is $x+5x+x$ equal to?

6. What is $x+2x+3x$ equal to?

7. James and John together have 24 peaches, and one has as many as the other. How many has each?

Let x denote the number which James has. Then, since they have an equal number, x will also denote the number which John has; and twice x will denote the number which both have, which is 24. If twice x is equal to 24, x will be equal to 24 divided by 2, which is 12: therefore each has 12 peaches. Hence we write, —

Let $x =$ number of peaches James has

Then $x + x = 2x = 24.$

Hence $x = \frac{24}{2} = 12.$

A verification is the operation of proving that the number found will satisfy the conditions of the question. Thus,

$$\overset{\text{James's apples.}}{12} + \overset{\text{John's apples.}}{12} = 24.$$

NOTE. — Let the following questions be analyzed, written, and verified in exactly the same manner as the above.

8. William and John together have 36 pears, and one has as many as the other. How many has each?

9. What number added to itself will make 20?

10. James and John are of the same age, and the sum of their ages is 32. What is the age of each?

11. Lucy and Ann are twins, and the sum of their ages is 16. What is the age of each?

12. What number is that which added to itself will make 30?

13. What number is that which added to itself will make 50?

14. Each of two boys received an equal sum of money at Christmas, and together they received 60 cents. How much did each receive?

15. What number added to itself will make 100?

16. John has as many pears as William. Together they have 72. How many has each?

17. What number added to itself will give a sum equal to 46?

18. Lucy and Ann have each a rosebush with the same number of buds on each. The buds on both number 46. How many on each?

Lesson II.

1. John and Charles together have 12 apples, and Charles has twice as many as John. How many has each?

Let x denote the number of apples which John has. Then, since Charles has twice as many, $2x$ will denote his share; and $x + 2x$, or $3x$, will denote the number which they both have, which is 12. If $3x$ is equal to 12, x will be equal to 12 divided by 3, which is 4: therefore John has 4 apples; and Charles, having twice as many, has 8. Hence we write, —

Let $\quad x =$ number of apples John has.

Then $\quad 2x =$ number of apples Charles has,

and $\quad x + 2x = 3x = 12$, the number both have.

Then $\quad x = \frac{12}{3} = 4$, the number John has,

and $\quad 2x = 2 \times 4 = 8$, the number Charles has.

Verification. $4 + 8 = 12$, the number both have.

2. William and John together have 48 quills, and William has twice as many as John. How many has each?

3. What number is that which added to twice itself will give a number equal to 60?

4. Charles's marbles added to John's make three times as many as Charles has. Together they have 51. How many has each?

NOTE. — Since Charles's marbles added to John's make three times as many as Charles has, Charles must have one third, and John two thirds, of the whole.

Let x denote the number which Charles has. Then $2x$ will denote the number which John has; and $x + 2x$, or $3x$, will denote what they both have, which is 51. Then, if $3x$ is equal to 51, x will be equal to 51 divided by 3, which is 17: therefore Charles has 17 marbles; and John, having twice as many, has 34. Hence we write, —

Let $x =$ number of Charles's marbles.

Then $2x =$ number of John's marbles,

and $3x = 51$, the number of both.

Then $x = \frac{51}{3} = 17$, Charles's marbles;

and $2x = 2 \times 17 = 34$, John's marbles.

5. What number added to twice itself will make 75?

6. What number added to twice itself will make 57?

7. What number added to twice itself will make 39?

8. What number added to twice itself will give 90?

9. John walks a certain distance on Tuesday, twice as far on Wednesday, and in the two days he walks 27 miles. How far did he walk each day?

10. Jane's bush has twice as many roses as Nancy's, and on both bushes there are 36. How many on each?

11. Samuel and James bought a ball for 48 cents. Samuel paid twice as much as James. What did each pay?

12. Divide 48 into two such parts that one shall be double the other.

13. Divide 66 into two such parts that one shall be double the other.

14. The sum of three equal numbers is 12. What are the numbers?

Let x denote one of the numbers. Then, since the numbers are equal, x will also denote each of the others; and x plus x plus x, or $3x$, will denote their sum, which is 12. Then, if $3x$ is equal to 12, x will be equal to 12 divided by 3, which is 4: therefore the numbers are 4, 4, and 4. Hence we write, —

Let $x =$ one of the equal numbers.

Then $x + x + x = 3x = 12,$

and $x = \frac{12}{3} = 4.$

VERIFICATION. $4 + 4 + 4 = 12.$

15. The sum of three equal numbers is 24. What are the numbers?

16. The sum of three equal numbers is 36. What are the numbers?

17. The sum of three equal numbers is 54. What are the numbers?

LESSON III.

1. What number is that which added to three times itself will make 48?

Let x denote the number. Then $3x$ will denote three times the number; and x plus $3x$, or $4x$, will denote the sum, which is 48. If $4x$ is equal to 48, x will be equal to 48 divided by 4, which is 12: therefore 12 is the required number. Hence we write, —

Let $x =$ the number.

Then $3x =$ three times the number,

and $x + 3x = 4x = 48$, the sum.

Then $x = \frac{48}{4} = 12$, the required number.

VERIFICATION. $12 + 3 \times 12 = 12 + 36 = 48$.

NOTE. — All similar questions may be solved by the same form of analysis.

2. What number added to four times itself will give 40?

3. What number added to five times itself will give 42?

4. What number added to six times itself will give 63?

5. What number added to seven times itself will give 88?

6. What number added to eight times itself will give 81?

7. What number added to nine times itself will give 100?

8. James and John together have 24 quills, and John has three times as many as James. How many has each?

9. William and Charles have 64 marbles, and Charles has seven times as many as William. How many has each?

10. James and John travel 96 miles, and James travels eleven times as far as John. How far does each travel?

11. The sum of the ages of a father and son is 84 years, and the father is three times as old as the son. What is the age of each?

12. There are two numbers of which the greater is seven times the less, and their sum is 72. What are the numbers?

13. The sum of four equal numbers is 64. What are the numbers?

14. The sum of six equal numbers is 54. What are the numbers?

15. James has 24 marbles. He loses a certain number, and then gives away seven times as many as he loses, which takes all he has. How many did he give away? Verify.

16. William has 36 cents, and divides them between his two brothers, James and Charles, giving one brother eight times as many as the other. How many does he give to each?

17. What is the sum of x and $3x$? Of x and $7x$? Of x and $5x$? Of x and $12x$?

Lesson IV.

1. If 1 apple costs 1 cent, what will a number of apples denoted by x cost?

Since 1 apple costs 1 cent, and since x denotes *any number* of apples, the cost of x apples will be as many cents as there are apples; that is, x cents.

2. If 1 apple costs 2 cents, what will x apples cost?

Since 1 apple costs 2 cents, and since x denotes the number of apples, the cost will be twice as many cents as there are apples; that is, $2x$ cents.

3. If 1 apple costs 3 cents, what will x apples cost?

4. If 1 lemon costs 4 cents, what will x lemons cost?

5. If 1 orange costs 6 cents, what will a number of oranges denoted by x cost?

6. Charles bought a certain number of lemons at 2 cents apiece, and as many oranges at 3 cents apiece, and paid in all 20 cents. How many did he buy of each?

Let x denote the number of lemons. Then, since he bought as many oranges as lemons, it will also denote the number of oranges. Since the lemons were 2 cents apiece, $2x$ will denote the cost of the lemons; and since the oranges were 3 cents apiece, $3x$ will denote the cost of the oranges; and $2x$ plus $3x$, or $5x$, will denote the cost of both, which is 20 cents. Now, since $5x$ cents are equal to 20 cents, x will be equal to 20 cents divided by 5 cents, which is 4: hence he bought 4 of each. Hence we write,—

Let $x =$ the number of lemons, or oranges.

Then $2x =$ the cost of the lemons,

and $3x =$ the cost of the oranges.

Hence $2x + 3x = 5x = 20$ cents $=$ the cost of lemons and oranges.

Hence $x = \dfrac{20 \text{ cents}}{5 \text{ cents}} = 4$, the number of each.

VERIFICATION. 4×2 cents $=$ 8 cents, cost of lemons.
4×3 cents $=$ 12 cents, cost of oranges.

20 cents, total cost.

7. A farmer bought a certain number of sheep at $4 apiece, and an equal number of lambs at $1 apiece, and the whole cost $60. How many did he buy of each?

8. Charles bought a certain number of apples at 1 cent apiece, and an equal number of oranges at 4 cents apiece, and paid 50 cents in all. How many did he buy of each?

9. James bought an equal number of apples, pears, and lemons. He paid 1 cent apiece for the apples, 2 cents apiece for the pears, and 3 cents apiece for the lemons, and paid 72 cents in all. How many did he buy of each? Verify.

10. A farmer bought an equal number of sheep, hogs, and calves, for which he paid $108. He paid $3 apiece for the sheep, $5 apiece for the hogs, and $4 apiece for the calves. How many did he buy of each?

11. A farmer sold an equal number of ducks, geese, and turkeys, for which he received 90 shillings. The ducks brought him 3 shillings apiece, the geese 5, and the turkeys 7. How many did he sell of each sort?

12. A tailor bought, for $100, two pieces of cloth, each of which contained an equal number of yards. For one piece he paid $3 a yard, and for the other $2 a yard. How many yards in each piece?

13. The sum of three numbers is 28. The second is twice the first, and the third twice the second. What are the numbers? Verify.

14. The sum of three numbers is 64. The second is three times the first, and the third four times the second. What are the numbers?

Lesson V.

1. If 1 yard of cloth costs x dollars, what will 2 yards cost?

Two yards of cloth will cost twice as much as one yard: therefore, if 1 yard of cloth costs x dollars, 2 yards will cost twice x dollars, or $2x$ dollars.

2. If 1 yard of cloth costs x dollars, what will 3 yards cost? Why?

3. If 1 orange costs x cents, what will 7 oranges cost? Why? 8 oranges?

4. Charles bought 3 lemons and 4 oranges, for which he paid 22 cents. He paid twice as much for an orange as for a lemon. What was the price of each?

Let x denote the price of a lemon. Then $2x$ will denote the price of an orange, $3x$ will denote the cost of 3 lemons, and $8x$ the cost of 4 oranges; hence $3x$ plus $8x$, or $11x$, will denote the cost of the lemons and oranges, which is 22 cents. If $11x$ is equal to 22 cents, x is equal to 22 cents divided by 11, which is 2 cents: therefore the price of 1 lemon is 2 cents; and that of 1 orange, 4 cents. Hence we write,—

Let $x =$ the price of 1 lemon.

Then $2x =$ the price of 1 orange,

and $3x + 8x = 11x = 22$ cents, cost of lemons and oranges.

Hence $x = \frac{22 \text{ cents}}{11} = 2$ cents, price of 1 lemon;

and $2x = 2 \times 2$ cents $= 4$ cents, price of 1 orange.

VERIFICATION. 3×2 cents $=$ 6 cents, cost of lemons.
4×4 cents $=$ 16 cents, cost of oranges.
22 cents, total cost.

5. James bought 8 apples and 3 oranges, for which he paid 20 cents. He paid as much for 1 orange as for 4 apples. What did he pay for one of each?

6. A farmer bought 3 calves and 7 pigs, for which he paid \$19. He paid four times as much for a calf as for a pig. What was the price of each?

7. James bought an apple, a peach, and a pear, for which he paid 6 cents. He paid twice as much for the peach as for the apple, and three times as much for the pear as for the apple. What was the price of each?

8. William bought an apple, a lemon, and an orange, for which he paid 24 cents. He paid twice as much for the lemon as for the apple, and three times as much for the orange as for the apple. What was the price of each?

9. A farmer sold 4 calves and 5 cows, for which he received \$120. He received as much for 1 cow as for 4 calves. What was the price of each?

10. Lucy bought 3 pears and 5 oranges, for which she paid 26 cents, giving twice as much for each orange as for each pear. What was the price of each?

11. Ann bought 2 skeins of silk, 3 pieces of tape, and a penknife, for which she paid 80 cents. She paid the same for the silk as for the tape, and as much for the penknife as for both. What was the cost of each?

12. James, John, and Charles are to divide 56 cents among them so that John shall have twice as many as James, and Charles twice as many as John. What is the share of each?

13. Put 54 apples into three baskets so that the second shall contain twice as many as the first, and the third as many as the first and second. How many will there be in each?

14. Divide 60 into four such parts that the second shall be double the first, the third double the second, and the fourth double the third. What are the numbers?

Lesson VI.

1. If $2x + x$ is equal to $3x$, what is $3x - x$ equal to?
Ans. (written) $3x - x = 2x$.

2. What is $4x - x$ equal to?
Ans. (written) $4x - x = 3x$.

3. What is $8x - 6x$ equal to?
Ans. (written) $8x - 6x = 2x$.

4. What is $12x - 9x$ equal to? *Ans.* $3x$.

5. What is $15x - 7x$ equal to? *Ans.* $8x$.

6. What is $17x - 13x$ equal to? *Ans.* $4x$.

7. Two men, who are 30 miles apart, travel towards each other, — one at the rate of 2 miles an hour, and the other at the rate of 3 miles an hour. How long before they will meet?

Let x denote the number of hours. Then, since the time multiplied by the rate will give the distance, $2x$ will denote the distance traveled by the first, and $3x$ the distance traveled by the second. But the sum of the distances is 30 miles: hence $2x + 3x = 5x = 30$ miles. If $5x$ is equal to 30, x is equal to 30 divided by 5, which is 6: hence they will meet in 6 hours. Hence we write, —

Let $x =$ the time in hours.
Then $2x =$ distance traveled by 1st,
and $3x =$ distance traveled by 2d.

By the conditions, $2x + 3x = 5x = 30$ miles, the distance apart.

Hence $x = \frac{30}{5} = 6$ hours.

VERIFICATION. 2×6 miles $= 12$ miles, distance traveled by 1st.
3×6 miles $= 18$ miles, distance traveled by 2d.

30 miles, whole distance.

8. Two persons are 10 miles apart, and are traveling in the same direction, — the first at the rate of 3 miles an hour, and the second at the rate of 5 miles. How long before the second will overtake the first?

Let x denote the time in hours. Then $3x$ will denote the distance traveled by the first in x hours, and $5x$ the distance traveled by the second. But when the second overtakes the first, he will have traveled 10 miles more than the first: hence $5x - 3x = 2x = 10$. If $2x$ is equal to 10, x is equal to 5: hence the second will overtake the first in 5 hours. Hence we write, —

Let $x =$ the time in hours.

Then $3x =$ the distance traveled by 1st,

and $5x =$ the distance traveled by 2d;

and $5x - 3x = 2x = 10$ hours,

or $x = \frac{10}{2} = 5$ hours.

VERIFICATION. 3×5 miles $= 15$ miles, distance traveled by 1st.
5×5 miles $= 25$ miles, distance traveled by 2d.

$25 - 15 = 10$ miles, distance apart.

9. A cistern holding 100 hogsheads is filled by two pipes. One discharges 8 hogsheads a minute, and the other 12. In what time will they fill the cistern?

10. A cistern holding 120 hogsheads is filled by 3 pipes. The first discharges 4 hogsheads in a minute, the second 7, and the third 1. In what time will they fill the cistern?

11. A cistern which holds 90 hogsheads is filled by a pipe which discharges 10 hogsheads a minute; but there is a

waste-pipe which loses 4 hogsheads a minute. How long will it take to fill the cistern?

12. Two pieces of cloth contain each an equal number of yards. The first cost $3 a yard, and the second $5, and both pieces cost $96. How many yards in each?

13. Two pieces of cloth contain each an equal number of yards. The first cost $7 a yard, and the second $5. The first cost $60 more than the second. How many yards in each piece?

14. John bought an equal number of oranges and lemons. The oranges cost him 5 cents apiece, and the lemons 3, and he paid 56 cents for the whole. How many did he buy of each kind?

15. Charles bought an equal number of oranges and lemons. The oranges cost him 5 cents apiece, and the lemons 3. He paid 14 cents more for the oranges than for the lemons. How many did he buy of each?

16. Two men work the same number of days. The one receives $1 a day, and the other $2. At the end of the time they receive $54. How long did they work?

Lesson VII.

1. John and Charles together have 25 cents, and Charles has 5 more than John. How many has each?

Let x denote the number which John has. Then $x + 5$ will denote the number which Charles has; and $x + x + 5$, or $2x + 5$, will be equal to 25, the number they both have. Since $2x + 5$ equals 25, $2x$ will be equal to $25 - 5$, or 20, and x will be equal to 20 divided by 2, or 10: therefore John has 10 cents, and Charles 15. Hence we write, —

Let $\qquad x =$ the number of John's cents.

Then $\qquad x + 5 =$ the number of Charles's cents,

and $\qquad x + x + 5 = 25$, the number they both have;

or $$2x + 5 = 25,$$

and $$2x = 25 - 5 = 20.$$

Hence $$x = \frac{20}{2} = 10, \text{ John's number,}$$

and $$x + 5 = 10 + 5 = 15, \text{ Charles's number.}$$

VERIFICATION. $\underset{\text{John's.}}{10} + \underset{\text{Charles's.}}{15} = 25$, the sum.

$\underset{\text{Charles's.}}{15} - \underset{\text{John's.}}{10} = 5$, the difference.

2. James and John have 30 marbles, and John has 4 more than James. How many has each?

3. William bought 60 oranges and lemons. There were 20 more lemons than oranges. How many were there of each sort?

4. A farmer has 20 more cows than calves. In all, he has 36. How many of each sort?

5. Lucy has 28 pieces of money in her purse, composed of cents and dimes. The cents exceed the dimes in number by 16. How many are there of each sort?

6. What number added to itself and to 9 will make 29?

7. What number added to twice itself and to 4 will make 25?

8. What number added to three times itself and to 12 will make 60?

9. John has five times as many marbles as Charles; and what they both have, added to 14, makes 44. How many has each?

10. There are three numbers, of which the second is twice the first, and the third twice the second; and when 9 is added to the sum, the result is 30. What are the numbers?

11. Divide 17 into two such parts that the second shall be two more than double the first. What are the parts?

12. Divide 40 into three such parts that the second shall be twice the first, and the third exceed six times the first by 4. What are the parts?

13. Charles has twice as many cents as James, and John has twice as many as Charles. If 7 be added to what they all have, the sum will be 28. How many has each?

14. Divide 15 into three such parts that the second shall be three times the first; the third twice the second, and 5 over. What are the numbers?

15. An orchard contains three kinds of trees, — apple, pear, and cherry. There are four times as many pear as apple trees; twice as many cherry as pear trees; and if 14 be added, the number will be 40. How many are there of each?

Lesson VIII.

1. John, after giving away 5 marbles, had 12 left. How many had he at first?

Let x denote the number. Then $x - 5$ will denote what he had left, which was equal to 12. Since x diminished by 5 is equal to 12, x will be equal to 12 increased by 5; that is, to 17: therefore he had 17 marbles. Hence we write, —

Let $x =$ the number he had at first.

Then $x - 5 = 12$, what he had left;

and $x = 12 + 5 = 17$, what he first had.

Verification. $17 - 5 = 12$, what were left.

2. Charles lost 6 marbles, and has 9 left. How many had he at first?

3. William gave 15 cents to John, and had 9 left. How many had he at first?

4. Ann plucked 8 buds from her rosebush, and there were 19 left. How many were there at first?

5. William took 27 cents from his purse, and there were 13 left. How many were there at first?

6. The sum of two numbers is 14, and their difference is 2. What are the numbers?

The difference of two numbers, added to the less, will give the greater. Let x denote the *less* number. Then $x + 2$ will denote the greater; and $x + x + 2$ will denote their sum, which is 14. Then $2x + 2$ equals 14; and $2x$ equals $14 - 2$, or 12: hence x equals 12 divided by 2, or 6: therefore the numbers are 6 and 8.

VERIFICATION. $8 + 6 = 14$, their sum;
and $8 - 6 = 2$, their difference.

7. The sum of two numbers is 18, and their difference 6. What are the numbers?

8. James and John have 26 marbles, and James has 4 more than John. How many has each?

9. Jane and Lucy have 16 books, and Lucy has 8 more than Jane. How many has each?

10. William bought an equal number of oranges and lemons. Charles took 5 lemons, after which William had but 25 of both sorts. How many did he buy of each?

11. Mary has an equal number of roses on each of two bushes. If she takes 4 from one bush, there will remain 24 on both. How many on each at first?

12. The sum of two numbers is 20, and their difference is 6. What are the numbers?

If x denotes the *greater* number, $x - 6$ will denote the less, and $x + x - 6$ will be equal to 20: hence $2x$ equals $20 + 6$, or 26; and x equals 26 divided by 2, equals 13: therefore the numbers are 13 and 7. Hence we write, —

Let $x =$ the greater.

Then $x - 6 =$ the less,

and $x + x - 6 = 20$, their sum.

Hence $2x = 20 + 6 = 26$;

or $x = \frac{26}{2} = 13$, and $13 - 6 = 7$.

VERIFICATION. $13 + 7 = 20$, and $13 - 7 = 6$.

13. The sum of the ages of a father and son is 60 years, and their difference is just half that number. What are their ages?

14. The sum of two numbers is 23, and the larger lacks 1 of being seven times the smaller. What are the numbers?

15. The sum of two numbers is 50. The larger is equal to ten times the less, minus 5. What are the numbers?

16. John has a certain number of oranges, and Charles has four times as many, less 7. Together they have 53. How many has each?

17. An orchard contains a certain number of apple trees, and three times as many cherry trees, less 6. The whole number is 30. How many of each sort?

LESSON IX.

1. If x denotes any number, and 1 be added to it, what will denote the sum? *Ans.* $x + 1$.

2. If 2 be added to x, what will denote the sum? If 3 be added, what? If 4 be added? etc.

3. If to John's marbles 1 marble be added, twice his number will be equal to 10. How many had he?

Let x denote the number. Then $x + 1$ will denote the number after 1 is added; and twice this number, or $2x + 2$, will be equal to 10. If

$2x + 2$ is equal to 10, $2x$ will be equal to $10 - 2$, or 8; or x will be equal to 4. Hence we write, —

Let $\quad x =$ the number of John's marbles.

Then $\quad x + 1 =$ the number after 1 is added,

and $\quad 2(x + 1) = 2x + 2 = 10.$

Hence $\quad 2x = 10 - 2,$

or $\quad x = \frac{8}{2} = 4.$

VERIFICATION. $\quad 2(4 + 1) = 2 \times 5 = 10.$

4. Write $x + 2$ multiplied by 3. *Ans.* $3(x + 2)$.
What is the product? *Ans.* $3x + 6$.

5. Write $x + 4$ multiplied by 5. *Ans.* $5(x + 4)$.
What is the product? *Ans.* $5x + 20$.

6. Write $x + 3$ multiplied by 4. *Ans.* $4(x + 3)$.
What is the product? *Ans.* $4x + 12$.

7. Lucy has a certain number of books. Her father gives her two more, when twice her number is equal to 14. How many has she?

8. Jane has a certain number of roses in blossom. Two more bloom, and then three times the number is equal to 15. How many were in blossom at first?

9. Jane has a certain number of handkerchiefs, and buys four more, when five times her number is equal to 45. How many had she at first?

10. John has one apple more than Charles, and three times John's added to what Charles has make 15. How many has each?

Let x denote Charles's apples. Then $x + 1$ will denote John's; and $x + 1$ multiplied by 3, added to x, or $3x + 3 + x$, will be equal to 15, what they both had: hence $4x + 3$ equals 15; and $4x$ equals $15 - 3$, or 12; and $x = 4$. Write and verify.

11. James has two marbles more than William, and twice his marbles plus twice William's are equal to 16. How many has each?

12. Divide 20 into two such parts that one part shall exceed the other by 4.

13. A fruit-basket contains apples, pears, and peaches. There are 2 more pears than apples, and twice as many peaches as pears. There are 22 in all. How many of each sort?

14. What is the sum of $x + 3x + 2(x + 1)$?

15. What is the sum of $2(x + 1) + 1(x + 1) + x$?

16. What is the sum of $x + 5(x + 8)$?

17. The sum of two numbers is 11, and the second is equal to twice the first plus 2. What are the numbers?

18. John bought 3 apples, 3 lemons, and 3 oranges, for which he paid 27 cents. He paid one cent more for a lemon than for an apple, and 1 cent more for an orange than for a lemon. What did he pay for each?

19. Lucy, Mary, and Ann have 15 cents. Mary has 1 more than Lucy, and Ann twice as many as Mary. How many has each?

Lesson X.

1. If x denote any number, and 1 be subtracted from it, what will denote the difference? *Ans.* $x - 1$.

If 2 be subtracted, what will denote the difference? If 3 be subtracted? 4? etc.

2. John has a certain number of marbles. If 1 be taken away, twice the remainder will be equal to 12. How many has he?

Let x denote the number. Then $x-1$ will denote the number after 1 is taken away; and twice this number, or $2(x-1)=2x-2$, will be equal to 12. If $2x$ diminished by 2 is equal to 12, $2x$ is equal to $12+2$, or 14: therefore x equals 14 divided by 2, or 7. Hence we write, —

Let $\qquad x = $ the number.

Then $\qquad x-1 = $ the number which remained,

and $\qquad 2(x-1) = 2x-2 = 12.$

Hence $\qquad 2x = 12+2$, or 14;

and $\qquad x = \frac{14}{2} = 7.$

VERIFICATION. $\qquad 2(7-1) = 14-2 = 12;$

also $2(7-1) = 2\times 6 = 12.$

3. Write 3 times $x-1$. *Ans.* $3(x-1)$.
What is the product equal to? *Ans.* $3x-3$.

4. Write 4 times $x-2$. *Ans.* $4(x-2)$.
What is the product equal to? *Ans.* $4x-8$.

5. Write 5 times $x-5$. *Ans.* $5(x-5)$.
What is the product equal to? *Ans.* $5x-25$.

6. If x denotes a certain number, will $x-1$ denote a greater, or less, number? How much less?

7. If $x-1$ is equal to 4, what will x be equal to?
Ans. $4+1$, or 5.

8. If $x-2$ is equal to 6, what is x equal to?

9. James and John together have 20 oranges. John has 6 less than James. How many has each?

10. A grocer sold 12 pounds of tea and coffee. If the tea be diminished by 3 pounds, and the remainder multiplied by 2, the product is the number of pounds of coffee. How many pounds of each?

11. Ann has a certain number of oranges. Jane has 1 less, and twice her number added to Ann's make 13. How many has each?

Let x denote the number of oranges which Ann has. Then $x-1$ will denote the number Jane has; and $x+2x-2$, or $3x-2$, will denote the number both have, which is 13. If $3x-2$ equals 13, $3x$ will be equal to $13+2$, or 15; and if $3x$ is equal to 15, x will be equal to 15 divided by 3, which is 5: therefore Ann has 5 oranges, and Jane 4. Hence we write, —

Let $x =$ the number Ann has.

Then $x-1 =$ the number Jane has,

and $2(x-1) = 2x-2 =$ twice what Jane has,

also $x+2x-2 = 3x-2 = 13.$

Hence $3x = 13+2 = 15;$

or $x = \frac{15}{3} = 5.$

VERIFICATION. $5-4=1$; and $2 \times 4 + 5 = 13$.

12. Charles and John have 20 cents, and John has 6 less than Charles. How many has each?

13. James has twice as many oranges as lemons in his basket, and if 5 be taken from the whole number, 19 will remain. How many had he of each?

14. A basket contains apples, peaches, and pears, 29 in all. If 1 be taken from the number of apples, the remainder will denote the number of peaches, and twice that remainder will denote the number of pears. How many are there of each sort?

15. If $2x-5$ equals 15, what is the value of x?

16. If $4x-5$ is equal to 11, what is the value of x?

17. If $5x-12$ is equal to 18, what is the value of x?

18. The sum of two numbers is 32, and the greater exceeds the less by 8. What are the numbers?

19. The sum of two numbers is 9. If the greater number be diminished by 5, and the remainder multiplied by 3, the product will be the less number. What are the numbers?

20. There are three numbers such that 1 taken from the first will give the second, the second multiplied by 3 will give the third, and their sum is equal to 26. What are the numbers?

21. John and Charles together have just 31 oranges. If 1 be taken from John's, and the remainder be multiplied by 5, the product will be equal to Charles's number. How many has each?

22. A basket is filled with apples, lemons, and oranges; in all, 26. The number of lemons exceed the number of apples by 2, and the number of oranges is double that of the lemons. How many are there of each?

Lesson XI.

1. John has a certain number of apples, the half of which is equal to 10. How many has he?

Let x denote the number of apples. Then x divided by 2 is equal to 10. If one half of x is equal to 10, twice one half of x, or x, is equal to twice 10, which is 20: hence x is equal to 20.

Note. — A similar analysis is applicable to any one of the fractional units. Let each question be solved according to the analysis.

2. John has a certain number of oranges, and one third of his number is 15. How many has he?

3. If one fifth of a number is 6, what is the number?

4. If one twelfth of a number is 9, what is the number?

5. What number added to one half of itself will give a sum equal to 12?

Denote the number by x. Then x plus one half of x equals 12. But x plus one half of x equals three halves of x: hence three halves of x equal 12. If three halves of x equal 12, one half of x equals one third of 12, or 4. If one half of x equals 4, x equals twice 4, or 8: therefore x equals 8. Hence we write, —

Let $x =$ the number.

Then $x + \frac{1}{2}x = \frac{3}{2}x = 12,$

and $\frac{1}{2}x = 4,$ or $x = 8.$

VERIFICATION. $8 + \frac{8}{2} = 8 + 4 = 12.$

6. What number added to one third of itself will give a sum equal to 12?

7. What number added to one fourth of itself will give a sum equal to 20?

8. What number added to a fifth of itself will make 24?

9. What number diminished by one half of itself will leave 4? Why?

10. What number diminished by one third of itself will leave 6?

11. James gave one seventh of his marbles to William, and then had 24 left. How many had he at first?

12. What number added to two thirds of itself will give a sum equal to 20?

13. What number diminished by three fourths of itself will leave 9?

14. What number added to five sevenths of itself will make 24?

15. What number diminished by seven eighths of itself will leave 4?

16. What number added to eight ninths of itself will make 34?

17. What number diminished by eleven twelfths of itself will leave 5?

18. Margaret gave nine tenths of her apples to her sister, and then had 6 left. How many had she at first?

19. What number added to 3 times one ninth of itself will give 72?

20. Henry had a certain number of cents. He lost one third of them, and had 15 left. How many had he at first?

CHAPTER I.

DEFINITIONS.

1. Quantity is anything which can be increased, diminished, and measured; as number, distance, weight, time, etc.

To measure a thing is to find how many times it contains some other thing of the same kind, taken as a standard. The assumed standard is called the **unit of measure.**

2. Mathematics is the science which treats of the measurement, properties, and relations of quantities.

In pure mathematics there are but eight kinds of quantity, and consequently but eight kinds of units; viz., units of number, units of currency, units of length, units of surface, units of volume, units of weight, units of time, and units of angular measure.

3. Algebra is a branch of mathematics in which the quantities considered are represented by letters, and the operations to be performed are indicated by signs.

4. The quantities employed in algebra are of two kinds, — **known** and **unknown.**

Known quantities are those whose values are given. They are generally represented by the leading letters of the alphabet; as, a, b, c, etc.

Unknown quantities are those whose values are required. They are generally represented by the final letters of the alphabet; as, x, y, z, etc.

When an unknown quantity becomes known, it is often denoted by the same letter, with one or more accents; as,

x', x'', x'''. These symbols are read, "x prime," "x second," "x third."

5. The **sign of addition** (+) is called **plus.** When placed between two quantities, it indicates that the second is to be added to the first. Thus, $a + b$ is read "a plus b," and indicates that b is to be added to a. If no sign is written, the sign + is understood.

The sign + is sometimes called the **positive sign**; and the quantities before which it is written are called **positive quantities**, or **additive quantities.**

6. The **sign of subtraction** (−) is called **minus.** When placed between two quantities, it indicates that the second is to be subtracted from the first. Thus, the expression $c - d$, read "c minus d," indicates that d is to be subtracted from c. If a stands for 6, and d for 4, then $a - d$ is equal to $6 - 4$, which is equal to 2.

The sign − is sometimes called the **negative sign**; and the quantities before which it is written are called **negative quantities**, or **subtractive quantities.**

7. The **sign of multiplication** (×) is read "multiplied by," or "multiplied into." When placed between two quantities, it indicates that the first is to be multiplied by the second. Thus, $a \times b$ indicates that a is to be multiplied by b. If a stands for 7, and b for 5, then $a \times b$ is equal to 7×5, which is equal to 35.

The multiplication of quantities is also indicated by simply writing the letters one after the other, and sometimes by placing a point between them. Thus, $a \times b$ signifies the same thing as ab or as $a \,.\, b$; $a \times b \times c$ signifies the same thing as abc or as $a \,.\, b \,.\, c$.

8. A **factor** is any one of the multipliers of a product. Factors are of two kinds, — **numeral** and **literal.** Thus, in the

expression $5abc$ there are four factors, — the numeral factor 5, and the three literal factors, a, b, and c.

9. The **sign of division** ($\div$) is read "divided by." When written between two quantities, it indicates that the first is to be divided by the second.

There are three signs used to denote division. Thus, $a \div b$, $\frac{a}{b}$, $a\underline{|b}$, denote that a is to be divided by b.

10. The **sign of equality** ($=$) is read "equal to." When written between two quantities, it indicates that they are equal to each other. Thus, the expression $a + b = c$ indicates that the sum of a and b is equal to c. If a stands for 3, and b for 5, c will be equal to 8.

11. The **sign of inequality** ($>$ or $<$) is read "greater than" or "less than." When placed between two quantities, it indicates that they are unequal, the greater one being placed at the opening of the sign. Thus, the expression $a > b$ indicates that a is greater than b, and the expression $c < d$ indicates that c is less than d.

12. The sign $\therefore$ means "therefore," or "consequently."

13. A **coefficient** is a number written before a quantity, to show how many times the quantity is taken additively. Thus, in the expression $a + a + a + a + a = 5a$, 5 is the coefficient of a.

A coefficient may be denoted either by a number or by a letter. Thus, $5x$ indicates that x is taken 5 times, and ax indicates that x is taken a times. If no coefficient is written, the coefficient 1 is understood. Thus, a is the same as $1a$.

14. An **exponent** is a number written at the right and a little above a quantity, to indicate how many times the quantity is taken as a factor. Thus,

$$a \times a \text{ is written } a^2,$$
$$a \times a \times a \quad " \quad a^3,$$
$$a \times a \times a \times a \quad " \quad a^4, \text{ etc.}$$

and the 2, 3, and 4 are exponents. The expressions are read, "a square," "a cube" or "a third," "a fourth;" and if we have a^m, in which a enters m times as a factor, it is read "a to the mth," or simply "a mth." The exponent 1 is generally omitted. Thus, a^1 is the same as a, each denoting that a enters but once as a factor.

15. A **power** is a product which arises from the multiplication of equal factors. Thus,

$$a \times a = a^2 = \text{the square, or second power, of } a.$$
$$a \times a \times a = a^3 = \text{the cube, or third power, of } a.$$
$$a \times a \times a \times a = a^4 = \text{the fourth power of } a.$$
$$a \times a \times \ldots . = a^m = \text{the } m\text{th power of } a.$$

16. A **root** of a quantity is one of the equal factors. The radical sign ($\sqrt{}$), when placed over a quantity, indicates that a root of that quantity is to be extracted. The root is indicated by a number written over the radical sign, called an **index**. When the index is 2, it is generally omitted. Thus,

$\sqrt[2]{a}$, or $\sqrt{a}$, indicates the square root of a.
$\sqrt[3]{a}$ indicates the cube root of a.
$\sqrt[4]{a}$ indicates the fourth root of a.
$\sqrt[m]{a}$ indicates the mth root of a.

17. An **algebraic expression** is a quantity written in algebraic language. Thus, $3a$ is the algebraic expression of three times the number denoted by a; $5a^2$, that of five times the square of a; $7a^3b^2$, that of seven times the cube of a multiplied by the square of b; $3a - 5b$, that of the differ-

ence between three times a and five times b; and $2a^2 - 3ab + 4b^2$, that of twice the square of a, diminished by three times the product of a by b, augmented by four times the square of b.

18. A **term** is an algebraic expression that can be written without the aid of either of the signs + or —. Thus, $3a$, $2ab$, $5a^2b^2$, are terms.

A term may be preceded by either of the signs + or —. In this case the sign is called the **sign of the term,** and is used to show the *sense* in which the term is taken. Thus $+3a$ shows that $3\ a$ is taken positively, and $-5a^2b^2$ shows that $5a^2b^2$ is taken negatively.

19. The **degree** of a term is the number of its literal factors. Thus, $3a$ is a term of the first degree, because it contains but one literal factor; $5a^2$ is a term of the second degree, because it contains two literal factors (the factors in this case are equal); $7a^3b$ is a term of the fourth degree, because it contains four literal factors.

The degree of a term is determined by the sum of the exponents of all its letters.

20. A **monomial** is a single term, unconnected with any other by the signs + or —. Thus, $3a^2$, $3b^3a$, are monomials.

21. A **polynomial** is a collection of terms connected by the signs + or —; as, $3a - 5$, or $2a^3 - 3b + 4b^2$.

22. A **binomial** is a polynomial of two terms; as, $a + b$, $3a^2 - c^2$, $6ab - c^2$.

23. A **trinomial** is a polynomial of three terms; as, $abc - a^3 + c^3$, $ab - gh - f$.

24. **Homogeneous terms** are those which are of the same degree. Thus, the terms abc, $-a^3$, $+c^3$, are homogeneous; as are the terms ab, $-gh$.

25. A polynomial is homogeneous when all its terms are homogeneous. Thus, the polynomial $abc - a^3 + c^3$ is homogeneous, but the polynomial $ab - gh - f$ is not homogeneous.

26. **Similar terms** are those which have a common unit; that is, have a common literal part. Thus, $7ab + 3ab - 2ab$ are similar terms, and so also are $4a^2b^2 - 2a^2b^2 - 3a^2b^2$; but the terms of the first polynomial and of the last are not similar.

27. The **vinculum,** ——; the **bar,** |; the **parentheses,** ( ); and the **brackets,** [ ], — are each used to connect two or more terms which are to be operated upon as a whole. Thus, each of the expressions

$$\overline{a+b+c} \times x, \quad \left.\begin{array}{r} a \\ +b \\ +c \end{array}\right| x, \quad (a+b+c) \times x, \quad [a+b+c] \times x,$$

indicates that the *sum* of a, b, and c, is to be multiplied by x.

28. The **reciprocal** of a quantity is 1 divided by that quantity. Thus,

$$\frac{1}{a}, \ \frac{1}{a+b}, \ \frac{c}{d}, \text{ are the reciprocals of } a, \ a+b, \ \frac{d}{c}.$$

29. The **numerical value** of an algebraic expression is the result obtained by assigning a numerical value to each letter, and then performing the operations indicated. Thus, the numerical value of the expression $ab + bc + d$, when $a = 1$, $b = 2$, $c = 3$, and $d = 4$, is $1 \times 2 + 2 \times 3 + 4 = 12$, by performing the indicated operations.

Exercises in Writing Algebraic Expressions.

1. Write a added to b. *Ans.* $a + b$.
2. Write b subtracted from a. *Ans.* $a - b$.

Write the following: —

3. Six times the square of a, minus twice the square of b.

4. Six times a multiplied by b, diminished by 5 times c cube multiplied by d.

5. Nine times a, multiplied by c plus d, diminished by 8 times b multiplied by d cube.

6. Five times a minus b, plus 6 times a cube into b cube.

7. Eight times a cube into d fourth, into c fourth, plus 9 times c cube into d fifth, minus 6 times a into b, into c square.

8. Fourteen times a plus b, multiplied by a minus b, plus 5 times a, into c plus d.

9. Six times a, into c plus d, minus 5 times b, into a plus c, minus 4 times a cube b square.

10. Write a, multiplied by c plus d, plus f minus g.

11. Write a divided by $b + c$ in three ways.

12. Write $a - b$ divided by $a + b$.

13. Write a polynomial of three terms, of four terms, of five, of six.

14. Write a homogeneous binomial of the first degree, of the second, of the third, fourth, fifth, sixth.

15. Write a homogeneous trinomial of the first degree, with its second and third terms negative; of the second degree; of the third; of the fourth.

16. Write in the same column, on the slate or blackboard, a monomial; a binomial; a trinomial; a polynomial of four terms, of five terms, of six terms, and of seven terms; and all of the *same degree.*

Exercises in Interpretation of Algebraic Language.

Find the numerical values of the following expressions when $a=1$, $b=2$, $c=3$, and $d=4$:—

1. $ab+bc$. *Ans.* 8.
2. $a+bc+d$. *Ans.* 11.
3. $ad+b-c$. *Ans.* 3.
4. $ab+bc-d$. *Ans.* 4.
5. $(a+b)c^2-d$. *Ans.* 23.
6. $(a+b)(d-b)$. *Ans.* 6.
7. $(ab+ad)c+d$. *Ans.* 22.
8. $(ab+c)(ad-a)$. *Ans.* 15.
9. $3a^2b^2-2(a+d+1)$. *Ans.* 0.
10. $\frac{a+c}{2}\times(a+d)$. *Ans.* 10.
11. $\frac{a^2+b^2+c^2}{7}\times\frac{a^3+b^3+c^3-d}{2}$. *Ans.* 32.
12. $\frac{ab^4-c-a^3}{6}\times\frac{4a^2-b+d^3}{33}$. *Ans.* 4.

Find the numerical values of the following expressions when $a=4$, $b=3$, $c=2$, and $d=1$:—

13. $\frac{a}{2}-\frac{b}{3}+c-d$. *Ans.* 2.
14. $5\left(\frac{ab}{3}-\frac{a-d}{3}\right)$. *Ans.* 15.
15. $[(a^2b+1)d]\div(a^2b+d)$. *Ans.* 1.
16. $4\left(abc-\frac{b^3}{9}\right)\times(30c^3-ab^3d^3)$. *Ans.* 11088.
17. $\frac{a+b+c}{a-b+d}+\frac{abcd}{ab}+\frac{4a^2+b^2-d^2}{bc+b}$. *Ans.* $14\frac{1}{2}$.
18. $\frac{15(a+d+b)}{3c^2}-\frac{a-c}{2}+\frac{3}{abd}\times a^3b^3c^3d^3$. *Ans.* 3465.

CHAPTER II.

FUNDAMENTAL OPERATIONS.

ADDITION.

30. Addition is the operation of finding the simplest equivalent expression for the aggregate of two or more algebraic quantities. Such expression is called their **sum**.

31. When the Terms are Similar and have Like Signs.

(1) What is the sum of a, $2a$, $3a$, and $4a$?

Take the sum of the coefficients, and annex the common unit or literal part. The first term (a) has a coefficient 1 understood (§ 13).

$$\begin{array}{r} +\ \ a \\ +\ 2a \\ +\ 3a \\ +\ 4a \\ \hline +10a \end{array}$$

(2) What is the sum of $2ab$, $3ab$, $6ab$, and ab?

NOTE. — When no sign is written, the sign + is understood (§ 5).

$$\begin{array}{r} 2ab \\ 3ab \\ 6ab \\ ab \\ \hline 12ab \end{array}$$

Add the following: —

(3) $$\begin{array}{r} a \\ a \\ \hline +2a \end{array}$$

(5) $$\begin{array}{r} 7ac \\ 5ac \\ \hline 12ac \end{array}$$

(7) $$\begin{array}{r} -3abc \\ -2abc \\ \hline -5abc \end{array}$$

(9) $$\begin{array}{r} -2adf \\ -6adf \\ \hline -8adf \end{array}$$

(4) $$\begin{array}{r} 8ab \\ 7ab \\ \hline 15ab \end{array}$$

(6) $$\begin{array}{r} +4abc \\ 3abc \\ \hline +7abc \end{array}$$

(8) $$\begin{array}{r} -3ad \\ -2ad \\ \hline -5ad \end{array}$$

(10) $$\begin{array}{r} -\ 9abd \\ -15abd \\ \hline -24abd \end{array}$$

Hence, when the terms are similar and have like signs,

Add the coefficients, and to their sum prefix the common sign. To this annex the common unit or literal part.

Exercises.

1. $9ab + ax$	2. $8ac^2 - 3b^2$	3. $15ab^3c^4 - 12abc^2$
$3ab + 3ax$	$7ac^2 - 8b^2$	$12ab^3c^4 - 15abc^2$
$12ab + 4ax$	$3ac^2 - 9b^2$	$ab^3c^4 - abc^2$

32. When the Terms are Similar and have Unlike Signs.

The signs + and − stand in direct opposition to each other.

If a merchant writes + before his gains, and − before his losses, at the end of the year the sum of the plus numbers will denote the gains, and the sum of the minus numbers the losses. If the gains exceed the losses, the *difference*, which is called the **algebraic sum,** will be plus; but if the losses exceed the gains, the algebraic sum will be minus.

(1) A merchant in trade gained $1500 in the first quarter of the year, $3000 in the second quarter, but lost $3000 in the third quarter, and $800 in the fourth. What was the result of the year's business?

1st quarter	+ 1500	3d quarter	− 3000
2d "	+ 3000	4th "	− 800
	+ 4500		− 3800

$+4500 - 3800 = +700$, or $700 gain.

(2) A merchant in trade gained $1000 in the first quarter, and $2000 the second quarter. In the third quarter he lost $1500, and in the fourth quarter $1800. What was the result of the year's business?

1st quarter	+ 1000	3d quarter	− 1500
2d "	+ 2000	4th "	− 1800
	+ 3000		− 3300

$+3000 - 3300 = -300$, or $300 loss.

(3) A merchant in the first half year gained a dollars, and lost b dollars. In the second half year he lost a dollars, and gained b dollars. What is the result of the year's business?

1st half year	$+a$	1st half year	$-b$
2d "	$-a$	2d "	$+b$
Result	0	Result.	0

Hence the algebraic sum of a positive and a negative quantify is their arithmetical difference, with the sign of the greater prefixed.

Add the following: —

$$\begin{array}{lr}(4) & 8ab \\ & 3ab \\ & -6ab \\ \hline & 5ab\end{array} \qquad \begin{array}{lr}(5) & 4acb^2 \\ & -8acb^2 \\ & acb^2 \\ \hline & -3acb^2\end{array} \qquad \begin{array}{lr}(6) & -4a^2b^2c^2 \\ & +6a^2b^2c^2 \\ & -2a^2b^2c^2 \\ \hline & 0\end{array}$$

Hence, when the terms are similar and have unlike signs,

Write the similar terms in the same column.

Add the coefficients of the additive terms and also the coefficients of the subtractive terms.

Take the difference of these sums, prefix the sign of the greater, and then annex the literal part or unit.

Exercises.

What is the sum of

1. $2a^2b^3-5a^2b^3+7a^2b^3+6a^2b^3-11a^2b^3$?

$$\begin{array}{r} 2a^2b^3 \\ -\ \ 5a^2b^3 \\ +\ \ 7a^2b^3 \\ +\ \ 6a^2b^3 \\ -11a^2b^3 \\ \hline -\ \ \ a^2b^3\end{array}$$

Having written the similar terms in the same column, we find the sum of the positive coefficients to be 15, and the sum of the negative coefficients to be -16. Their difference is -1: hence the sum is $-a^2b^3$.

2. $3a^2b + 5a^2b - 3a^2b + 4a^2b - 6a^2b - a^2b$? *Ans.* $2a^2b$.

3. $12a^3bc^2 - 4a^3bc^2 + 6a^3bc^2 - 8a^3bc^2 + 11a^3bc^2$? *Ans.* $17a^3bc^2$.

4. $4a^2b - 8a^2b - 9a^2b + 11a^2b$? *Ans.* $-2a^2b$.

5. $7abc^2 - abc^2 - 7abc^2 + 8abc^2 + 6abc^2$? *Ans.* $13abc^2$.

6. $9cb^3 - 5cb^3 - 8ac^2 + 20cb^3 + 9ac^2 - 24cb^3$? *Ans.* $+ac^2$.

33. To Add any Algebraic Quantities.

(1) What is the sum of $3a$, $5b$, and $-2c$?

Write the quantities thus: $3a + 5b - 2c$, which indicates their sum, as the *terms* are *dissimilar;* that is, have no common unit.

(2) Let it be required to find the sum of the quantities

$$\begin{array}{rrr}
2a^2 & -\,4ab & \\
3a^2 & -\,3ab & +\,b^2 \\
 & 2ab & -\,5b^2 \\
\hline
5a^2 & -\,5ab & -\,4b^2
\end{array}$$

From the preceding examples we have, for the addition of algebraic quantities, the following rule:—

Write the quantities to be added, placing similar terms in the same column, and giving to each its proper sign.

Add each column separately, and then annex the dissimilar terms with their proper signs.

Exercises.

1. Add the polynomials

$3a^2 - 2b^2 - 4ab$, $5a^2 - b^2 + 2ab$, and $3ab - 3c^2 - 2b^2$.

$$\begin{array}{rrrr}
\cancel{3a^2} & -\,\cancel{4ab} & -\,\cancel{2b^2} & \\
\cancel{5a^2} & +\,\cancel{2ab} & -\,\cancel{b^2} & \\
 & +\,\cancel{3ab} & -\,\cancel{2b^2} & -\,\cancel{3c^2} \\
\hline
8a^2 & +\,ab & -\,5b^2 & -\,3c^2
\end{array}$$

The term $3a^2$ being similar to $5a^2$, we write $8a^2$ for the result of the reduction of these two terms, at the same time slightly crossing them, as in the first term.

Passing then to the term $-4ab$, which is similar to $+2ab$ and $+3ab$, the three reduce to $+ab$, which is placed after $8a^2$, and the terms crossed like the

first term. Passing then to the terms involving b^2, we find their sum to be $-5b^2$, after which we write $-3c^2$.

NOTE. — The marks are drawn across the terms, that none of them may be overlooked or omitted.

2.
$$\begin{array}{r} 7abc+9ax \\ -3abc-3ax \\ \hline 4abc+6ax \end{array}$$

3.
$$\begin{array}{r} 8ax+3b \\ 5ax-9b \\ \hline 13ax-6b \end{array}$$

4.
$$\begin{array}{r} 12a-\ \ 6c \\ -\ \ 3a-\ \ 9c \\ \hline 9a-15c \end{array}$$

5. If $a=5$, $b=4$, $c=2$, $x=1$, what are the numerical values of the several sums above found?

6.
$$\begin{array}{r} 9a+f \\ -6a+g \\ -2a- \\ \hline \end{array}$$

7.
$$\begin{array}{r} 6ax-\ \ 8ac \\ -7ax-\ \ 9ac \\ ax+17ac \\ \hline \end{array}$$

8.
$$\begin{array}{r} 3af+\ \ g+m \\ ag-3af-m \\ ab-\ ag+3g \\ \hline \end{array}$$

9.
$$\begin{array}{r} 7x+3ab+3c \\ -3x-3ab-5c \\ 5x-9ab-9c \\ \hline \end{array}$$

10.
$$\begin{array}{r} 8x^2+\ \ 9acx+13a^2b^2c^2 \\ -7x^2-13acx+14a^2b^2c^2 \\ -4x^2+\ \ 4acx-20a^2b^2c^2 \\ \hline \end{array}$$

11.
$$\begin{array}{r} 22h-3c-7f+3g \\ -\ \ 3h+8c-2f-9g+5x \\ \hline \end{array}$$

12.
$$\begin{array}{r} 19ah^2+3a^3b^4-8ax^3 \\ -17ah^2-9a^3b^4+9ax^3 \\ \hline \end{array}$$

13.
$$\begin{array}{r} 7x-9y+5z+3-\ \ g \\ -\ \ x-3y\qquad\ -8-\ \ g \\ -\ \ x+\ \ y-3z+1+7g \\ -2x+6y+3z-1-\ \ g \\ \hline \end{array}$$

14.
$$\begin{array}{r} 8a+\ \ b\qquad\qquad \\ 2a-\ \ b+\ \ c\qquad \\ -3a+\ \ b\qquad+2d \\ -6b-3c+3d \\ \hline \end{array}$$

15. Add $-b+3c-d-115e+6f-5g$,
$3b-2c-3d-e+27f$, $5c-8d+3f-7g$,
$-7b-6c+17d+9e-5f+11g$,
$-3b-5d-2e+6-9g+h$.

Ans. $-8b-109e+37f-10g+h$.

16. Add the polynomials $7a^2b - 3abc - 8b^2c - 9c^3 + cd^2$,
$8abc - 5a^2b + 3c^3 - 4b^2c + cd^2$,
and $4a^2b - 8c^3 + 9b^2c - 3d^3$.
Ans. $6a^2b + 5abc - 3b^2c - 14c^3 + 2cd^2 - 3d^3$.

17. What is the sum of $5a^2bc + 6bx - 4af$,
$-3a^2bc - 6bx + 14af$, $-af + 9bx + 2a^2bc$,
$+6af - 8bx + 6a^2bc$? *Ans.* $10a^2bc + bx + 15af$.

18. What is the sum of $a^2n^2 + 3a^3m + b$,
$-6a^2n^2 - 6a^3m - b$, $+9b - 9a^3m - 5a^2n^2$?
Ans. $-10a^2n^2 - 12a^3m + 9b$.

19. What is the sum of $4a^3b^2c - 16a^4x - 9ax^3d$,
$+6a^3b^2c - 6ax^3d + 17a^4x$, $+16ax^3d - a^4x - 9a^3b^2c$?
Ans. $a^3b^2c + ax^3d$.

20. What is the sum of $-7g + 3b + 4g - 2b + 3g - 3b$
$+2b$? *Ans.* 0.

21. What is the sum of $ab + 3xy - m - n$,
$-6xy - 3m + 11n + cd$, $+3xy + 4m - 10n + fg$?
Ans. $ab + cd + fg$.

22. What is the sum of $4xy + n + 6ax + 9am$,
$-6xy + 6n - 6ax - 8am$, $2xy - 7n + ax - am$?
Ans. $+ax$.

23.
$$\begin{array}{r} 2(a+b) \\ 3(a+b) \\ 2(a+b) \\ \hline 7(a+b) \end{array}$$

24.
$$\begin{array}{r} 5(a^2-c^2) \\ -4(a^2-c^2) \\ -1(a^2-c^2) \\ \hline \end{array}$$

25.
$$\begin{array}{r} 9(c^3-af^3) \\ 7(c^3-af^3) \\ -10(c^3-af^3) \\ \hline 6(c^3-af^3) \end{array}$$

NOTE. — The quantity within the parentheses must be taken as a *whole* (§ 27). In Exercise 23 the *sum* of a and b, indicated by $(a + b)$, is the unit; in Exercise 24 the *difference* of a^2 and c^2, indicated by $(a^2 - c^2)$, is the unit.

26. Add $3a(g^2-h^2)-2a(g^2-h^2)+4a(g^2-h^2)$
$+8a(g^2-h^2)-2a(g^2-h^2)$. *Ans.* $11a(g^2-h^2)$.

27. Add $3c(a^2c-b^2)-9c(a^2c-b^2)-7c(a^2c-b^2)$
$+15c(a^2c-b^2)+c(a^2c-b^2)$. *Ans.* $3c(a^2c-b^2)$.

34. In algebra the term "add" does not always, as in arithmetic, convey the idea of augmentation; nor the term "sum" the idea of a number numerically greater than any of the numbers added: for, if to a we add $-b$, we have $a-b$, which is, arithmetically speaking, a difference between the number of units expressed by a and the number of units expressed by b; consequently this result is numerically less than a. To distinguish this sum from an arithmetical sum, it is called the **algebraic sum.**

SUBTRACTION.

35. Subtraction is the operation of finding the difference between two algebraic quantities.

36. The **subtrahend** is the quantity to be subtracted; the **minuend,** the quantity from which it is taken.

37. The **difference** of two quantities is such a quantity as, added to the subtrahend, will give a sum equal to the minuend.

(1) From $17a$ take $6a$.

In this example, $17a$ is the minuend, and $6a$ the subtrahend. The difference is $11a$, *because* $11a$ added to $6a$ gives $17a$.

$$\begin{array}{r} 17a \\ 6a \\ \hline 11a \end{array}$$

NOTE. — The difference may be expressed by writing the quantities thus: $17a-6a=11a$, in which the sign of the subtrahend is changed from $+$ to $-$.

(2) From $15x$ take $-9x$.

The difference, or remainder, is such a quantity as, being added to the subtrahend $(-9x)$, will give the minuend $(15x)$. That quantity is $24x$, and may be found by simply *changing the sign* of the subtrahend, and adding: whence we may write, $15x-(-9x)=24x$.

$$\begin{array}{r} 15x \\ -\ 9x \\ \hline 24x \end{array}$$

(3) From $10ax$ take $a-b$.

The difference, or remainder, is such a quantity as, added to $a-b$, will give the minuend ($10ax$). What is that quantity?

If you change the signs of both terms of the subtrahend, and add, you have $10ax-a+b$. Is this the true remainder? Certainly: for, if you add the remainder to the subtrahend ($a-b$), you obtain the minuend ($10ax$).

$$\begin{array}{r} 10ax \qquad\quad \\ +a-b \\ \hline 10ax-a+b \\ +a-b \\ \hline 10ax \qquad\quad \end{array}$$

It is plain, that if you change the signs of all the terms of the subtrahend, and then add them to the minuend, and to this result add the given subtrahend, the last sum can be no other than the given minuend: hence the *first* result is the true difference, or remainder (§ 37).

From the preceding examples we have, for the subtraction of algebraic quantities, the following rule: —

Write the terms of the subtrahend under those of the minuend, placing similar terms in the same column.

Conceive the signs of all the terms of the subtrahend to be changed from $+$ *to* $-$ *or from* $-$ *to* $+$, *and then proceed as in addition.*

Exercises.

Subtract the following: —

1. $\begin{array}{r} 3ab \\ 2ab \\ \hline ab \end{array}$

2. $\begin{array}{r} 6ax \\ 3ax \\ \hline 3ax \end{array}$

3. $\begin{array}{r} 9abc \\ 7abc \\ \hline 2abc \end{array}$

4. $\begin{array}{r} 16a^2b^2c \\ 9a^2b^2c \\ \hline 7a^2b^2c \end{array}$

5. $\begin{array}{r} 17a^3b^3c \\ 3a^3b^3c \\ \hline 14a^3b^3c \end{array}$

6. $\begin{array}{r} 24a^2b^2x \\ 7a^2b^2x \\ \hline 17a^2b^2x \end{array}$

7. $\begin{array}{r} 3ax \\ 8c \\ \hline 3ax-8c \end{array}$

8. $\begin{array}{r} 4abx \\ 9ac \\ \hline 4abx-9ac \end{array}$

9. $\begin{array}{r} 2am \\ ax \\ \hline 2am-ax \end{array}$

10. From $9a^2b^2$ take $3a^2b^2$. *Ans.* $6a^2b^2$.

11. From $16a^2xy$ take $-15a^2xy$. *Ans.* $31a^2xy$.

12. From $12a^4y^3$ take $8a^4y^3$. *Ans.* $4a^4y^3$.

13. From $19a^5x^8y$ take $-18a^5x^8y$. *Ans.* $37a^5x^8y$.

14. From $3a^2b^3$ take $3a^3b^2$. *Ans.* $3a^2b^3-3a^3b^2$.

15. From $7a^2b^4$ take $6a^4b^2$. *Ans.* $7a^2b^4-6a^4b^2$.

16. From $3ab^2$ take a^2b^5. *Ans.* $3ab^2-a^2b^5$.

17. From x^2y take y^2x. *Ans.* x^2y-y^2x.

18. From $3x^2y^3$ take xy. *Ans.* $3x^2y^3-xy$.

19. From $8a^2y^3x$ take xyz. *Ans.* $8a^2y^3x-xyz$.

20. From $9a^2b^2$ take $-3a^2b^2$. *Ans.* $12a^2b^2$.

21. From $14a^2y^2$ take $-20a^2y^2$. *Ans.* $34a^2y^2$.

22. From $-24a^4b^5$ take $16a^4b^5$. *Ans.* $-40a^4b^5$.

23. From $-13x^2y^4$ take $-14x^2y^4$. *Ans.* x^2y^4.

24. From $-47a^3x^2y$ take $-5a^3x^2y$. *Ans.* $-42a^3x^2y$.

25. From $-94a^2x^2$ take $3a^2x^2$. *Ans.* $-97a^2x^2$.

26. From $a+x^2$ take $-y^3$. *Ans.* $a+x^2+y^3$.

27. From a^3+b^3 take $-a^3-b^3$. *Ans.* $2a^3+2b^3$.

28. From $-16a^2x^3y$ take $-19a^2x^3y$. *Ans.* $+3a^2x^3y$.

29. From a^2-x^2 take a^2+x^2. *Ans.* $-2x^2$.

For Review.

Note. — Exercise 2 is the same as Exercise 1, with the signs of the subtrahend changed.

Subtract the following: —

1.
$$\begin{array}{r} 6ac-5ab+c^2 \\ 3ac+3ab+7c \\ \hline 3ac-8ab+c^2-7c \end{array}$$

2.
$$\begin{array}{r} 6ac-5ab+c^2 \\ -3ac-3ab-7c \\ \hline 3ac-8ab+c^2-7c \end{array}$$

3.
$$\begin{array}{r} 6ax - a + 3b^2 \\ 9ax - x + b^2 \\ \hline -3ax - a + x + 2b^2 \end{array}$$

5.
$$\begin{array}{r} 5a^3 - 4a^2b + 3b^2c \\ -2a^3 + 3a^2b - 8b^2c \\ \hline 7a^3 - 7a^2b + 11b^2c \end{array}$$

4.
$$\begin{array}{l} 6yx - 3x^2 + 5b \\ yx - 3 + a \\ \hline 5yx - 3x^2 + 3 + 5b - a \end{array}$$

6.
$$\begin{array}{r} 4ab - cd + 3a^2 \\ 5ab - 4cd + 3a^2 + 5b^2 \\ \hline -ab + 3cd - 5b^2 \end{array}$$

7. From $a + 8$ take $c - 5$. *Ans.* $a - c + 13$.

8. From $6a^2 - 15$ take $9a^2 + 30$. *Ans.* $-3a^2 - 45$.

9. From $6xy - 8a^2c^3$ take $-7xy - a^2c^3$. *Ans.* $13xy - 7a^2c^3$.

10. From $a + c$ take $-a - c$. *Ans.* $2a + 2c$.

11. From $4(a + b)$ take $2(a + b)$. *Ans.* $2(a + b)$.

12. From $3(a + x)$ take $(a + x)$. *Ans.* $2(a + x)$.

13. From $9(a^2 - x^2)$ take $-2(a^2 - x^2)$. *Ans.* $11(a^2 - x^2)$.

14. From $6a^2 - 15b^2$ take $-3a^2 + 9b^2$. *Ans.* $9a^2 - 24b^2$.

15. From $3a^m - 2b^n$ take $a^m - 2b^n$. *Ans.* $2a^m$.

16. From $9c^2m^2 - 4$ take $4 - 7c^2m^2$. *Ans.* $16c^2m^2 - 8$.

17. From $6am + y$ take $3am - x$. *Ans.* $3am + x + y$.

18. From $3ax$ take $3ax - y$. *Ans.* $+y$.

19. From $-7f + 3m - 8x$ take $-6f - 5m - 2x + 3d + 8$.
Ans. $-f + 8m - 6x - 3d - 8$.

20. From $-a - 5b + 7c + d$ take $4b - c + 2d + 2k$.
Ans. $-a - 9b + 8c - d - 2k$.

21. From $-3a + b - 8c + 7e - 5f + 3h - 7x - 13y$ take $k + 2a - 9c + 8e - 7x + 7f - y - 3l - k$.
Ans. $-5a + b + c - e - 12f + 3h - 12y + 3l$.

22. From $2x - 4a - 2b + 5$ take $8 - 5b + a + 6x$.
Ans. $-4x - 5a + 3b - 3$.

23. From $3a+b+c-d-10$ take $c+2a-d$.
Ans. $a+b-10$.

24. From $3a+b+c-d-10$ take $b-19+3a$.
Ans. $c-d+9$.

25. From $a^3+3b^2c+ab^2-abc$ take b^3+ab^2-abc.
Ans. $a^3+3b^2c-b^3$.

26. From $12x+6a-4b+40$ take $4b-3a+4x+6d-10$.
Ans. $8x+9a-8b-6d+50$.

27. From $2x-3a+4b+6c-50$ take $9a+x+6b-6c-40$.
Ans. $x-12a-2b+12c-10$.

28. From $6a-4b-12c+12x$ take $2x-8a+4b-6c$.
Ans. $14a-8b-6c+10x$.

38. In algebra the term "difference" does not always, as in arithmetic, denote a number less than the minuend: for, if from a we subtract $-b$, the remainder will be $a+b$; and this is numerically greater than a. We distinguish between the two cases by calling this result the **algebraic difference.**

39. When a polynomial is to be subtracted from an algebraic expression, we inclose it in a parenthesis, place the minus sign before it, and then write it after the minuend. Thus, the expression $6a^2-(3ab-2b^2+2bc)$ indicates that the polynomial $3ab-2b^2+2bc$ is to be taken from $6a^2$. Performing the indicated operations by the rule for subtraction, we have the equivalent expression $6a^2-3ab+2b^2-2bc$.

The last expression may be changed to the former by changing the signs of the last three terms, inclosing them in a parenthesis, and prefixing the sign $-$. Thus,

$$6a^2-3ab+2b^2-2bc=6a^2-(3ab-2b^2+2bc).$$

In like manner any polynomial may be transformed, as indicated below.

$$7a^3 - 8a^2b - 4b^2c + 6b^3 = 7a^3 - (8a^2b + 4b^2c - 6b^3)$$
$$= 7a^3 - 8a^2b - (4b^2c - 6b^3).$$
$$8a^3 - 7b^2 + c - d = 8a^3 - (7b^2 - c + d)$$
$$= 8a^3 - 7b^2 - (-c + d).$$
$$9b^3 - a + 3a^2 - d = 9b^3 - (a - 3a^2 + d)$$
$$= 9b^3 - a - (-3a^2 + d).$$

NOTE.—The sign of every term is changed when it is placed within a parenthesis which has the minus sign before it, and also when it is brought out of such parenthesis.

40. From the preceding principles we have

$$a - (+b) = a - b,$$

and
$$a - (-b) = a + b.$$

The sign immediately preceding b is called the **sign of the quantity**; the sign preceding the parenthesis is called the **sign of operation**; and the sign resulting from the combination of the signs is called the **essential sign.**

When the sign of operation is different from the sign of the quantity, the essential sign will be $-$; when the sign of operation is the same as the sign of the quantity, the essential sign will be $+$.

MULTIPLICATION.

41. Multiplication is the operation of finding the product of two quantities.

The **multiplicand** is the quantity to be multiplied; the **multiplier** is that by which it is multiplied; and the **product** is the result. The multiplier and multiplicand are called **factors** of the product.

Exercises.

1. If a man earns a dollars in 1 day, how much will he earn in 6 days?

In 6 days he will earn six times as much as in 1 day. If he earns a dollars in 1 day, in 6 days he will earn $6a$ dollars.

2. If 1 hat costs d dollars, what will 9 hats cost?

Ans. $9d$ dollars.

3. If 1 yard of cloth costs c dollars, what will 10 yards cost?

Ans. $10c$ dollars.

4. If 1 cravat costs b cents, what will 40 cost?

Ans. $40b$ cents.

5. If 1 pair of gloves costs b cents, what will a pairs cost?

If 1 pair of gloves costs b cents, a pairs will cost as many times b cents as there are units in a; that is, b taken a times, or ab, which denotes the *product* of b by a or of a by b.

6. If a man's income is $3a$ dollars a week, how much will he receive in $4b$ weeks?

$$3a \times 4b = 12ab.$$

If we suppose $a = 4$ dollars, and $b = 3$ weeks, the product will be 144 dollars.

NOTE. — It is proved in arithmetic (Davies' "Standard Arithmetic," § 50) that the product is not altered by changing the arrangement of the factors; that is, $12ab = a \times b \times 12 = b \times a \times 12 = a \times 12 \times b$.

42. To find the Product of Two Positive Monomials.

Multiply $3a^2b^2$ by $2a^2b$.

We write,

$$3a^2b^2 \times 2a^2b = 3 \times 2 \times a^2 \times a^2 \times b^2 \times b = 3 \times 2aaaabbb;$$

in which a is a factor 4 times, and b a factor 3 times: hence (§ 14)

$$3a^2b^2 \times 2a^2b = 3 \times 2a^4b^3 = 6a^4b^3,$$

in which *we multiply the coefficients together, and add the exponents of the like letters.*

The product of any two positive monomials may be found in like manner. Hence the rule: —

Multiply the coefficients together, for a new coefficient.

Write after this coefficient all the letters in both monomials,

giving to each letter an exponent equal to the sum of its exponents in the two factors.

Exercises.

1. $8a^2bc^2 \times 7abd^2 = 56a^3b^2c^2d^2$.

2. $21a^3b^2cd \times 8abc^3 = 168a^4b^3c^4d$.

3. $4abc \times 7df = 28abcdf$.

Multiply the following: —

4.
$$\begin{array}{r} 3a^2b \\ 2a^2b \\ \hline 6a^4b^2 \end{array}$$

6.
$$\begin{array}{r} 6xyz \\ ay^2z \\ \hline 6axy^3z^2 \end{array}$$

8.
$$\begin{array}{r} 3ab^2c^3 \\ 9a^2b^3c \\ \hline 27a^3b^5c^4 \end{array}$$

5.
$$\begin{array}{r} 12a^2x \\ 12x^2y \\ \hline 144a^2x^3y \end{array}$$

7.
$$\begin{array}{r} a^2xy \\ 2xy^2 \\ \hline 2a^2x^2y^3 \end{array}$$

9.
$$\begin{array}{r} 87ax^2y \\ 3b^3x^4y^3 \\ \hline 261ab^3x^6y^4 \end{array}$$

10. $5a^3b^2x^2$ by $6c^5x^6$. *Ans.* $30a^3b^2c^5x^8$.

11. $10a^4b^5c^8$ by $7acd$. *Ans.* $70a^5b^5c^9d$.

12. $36a^8b^7c^6d^5$ by $20ab^2c^3d^4$. *Ans.* $720a^9b^9c^9d^9$.

13. $5a^m$ by $3ab^n$. *Ans.* $15a^{m+1}b^n$.

14. $3a^mb^3$ by $6a^2b^n$. *Ans.* $18a^{m+2}b^{n+3}$.

15. $6a^mb^n$ by $9a^5b^7$. *Ans.* $54a^{m+5}b^{n+7}$.

16. $5a^mb^n$ by $2a^pb^q$. *Ans.* $10a^{m+p}b^{n+q}$.

17. $5a^mb^2c^2$ by $2ab^nc$. *Ans.* $10a^{m+1}b^{n+2}c^3$.

18. $6a^2b^mc^n$ by $3a^3b^2c^2$. *Ans.* $18a^5b^{m+2}c^{n+2}$.

19. $20a^5b^5cd$ by $12a^2x^2y$. *Ans.* $240a^7b^5cdx^2y$.

20. $14a^4b^6d^4y$ by $20a^3c^2x^2y$. *Ans.* $280a^7b^6c^2d^4x^2y^2$.

21. $8a^3b^3y^4$ by $7a^4bxy^5$. *Ans.* $56a^7b^4xy^9$.

22. $75axyz$ by $5a^5bcdx^2y^2$. *Ans.* $375a^6bcdx^3y^3z$.

23. $64a^3m^5x^4yz$ by $8ab^2c^3$. *Ans.* $512a^4b^2c^3m^5x^4yz$.

24. $9a^2b^2c^2d^3$ by $12a^3b^4c^6$. *Ans.* $108a^5b^6c^8d^3$.

25. $216ab^7c^3d^8$ by $3a^3b^2c^5$. *Ans.* $648a^4b^9c^8d^8$.

26. $70a^8b^7c^4d^2fx$ by $12a^7b^5c^3dx^2y^3$. *Ans.* $840a^{15}b^{12}c^7d^3fx^3y^3$.

43. Multiplication of Polynomials.

(1) Multiply $a - b$ by c.

It is required to take the *difference* between a and b, c times; or to take c, $a - b$ times.

As we cannot subtract b from a, we begin by taking a, c times, which is ac; but this product is too large by b taken c times, which is bc: hence the true product is $ac - bc$.

If a, b, and c denote numbers, as $a = 8$, $b = 3$, and $c = 7$, the operation may be written in figures.

$$\begin{array}{c} a - b \\ \underline{\quad c \quad} \\ ac - bc \end{array}$$

$$\begin{array}{r} 8 - 3 = 5 \\ \underline{7 \qquad\quad\;\; 7} \\ 56 - 21 = 35 \end{array}$$

(2) Multiply $a - b$ by $c - d$.

It is required to take $a - b$ as many times as there are units in $c - d$.

If we take $a - b$ c times, we have $ac - bc$; but this product is too large by $a - b$ taken d times. $a - b$ taken d times is $ad - db$. Subtracting this product from the preceding by changing the signs of its terms (§ 37, rule), we have $(a - b) + (a - c) = ab - bc - ad + bd$.

$$\begin{array}{l} a - b \\ \underline{c - d} \\ ac - bc \\ \underline{\qquad - ad + bd} \\ ac - bc - ad + bd \end{array}$$

$$\begin{array}{l} 8 - 3 \quad = 5 \\ \underline{7 - 2 \quad = 5} \\ 56 - 21 \\ \underline{\quad - 16 + 6} \\ 56 - 37 + 6 = 25 \end{array}$$

From the preceding examples we deduce the following rule: —

When the factors have like signs, the sign of their product will be $+$.

When the factors have unlike signs, the sign of their product will be $-$.

Therefore we say in algebraic language that + multiplied by +, or − multiplied by −, gives +; − multiplied by +, or + multiplied by −, gives −.

Hence for the multiplication of polynomials we have the following rule: —

Multiply every term of the multiplicand by each term of the multiplier, observing that like signs give +, and unlike signs −; then reduce the result to its simplest form.

Exercises.

NOTE. — All the terms in the exercises below are positive.

Multiply the following: —

1. $3a^2 + 4ab + b^2$ by $2a + 5b$.

$$\begin{array}{llll}
3a^2 + & 4ab & + b^2 & \\
2a + & 5b & & \\
\hline
6a^3 + & 8a^2b + & 2ab^2 & \\
 & +15a^2b + & 20ab^2 & + 5b^3 \\
\hline
6a^3 + & 23a^2b + & 22ab^2 & + 5b^3
\end{array}$$

44. NOTE. — It will be found convenient to arrange the terms of the polynomials with reference to the ascending or descending powers of some letter: that is, to write them down so that the highest or lowest power of that letter shall enter the first term; the next highest or lowest, the second term; and so on to the last term.

The letter with reference to which the arrangement is made is called the **leading letter**. In the above example the leading letter is a. The leading letter of the product will always be the same as that of the factors.

2. $x^2 + 2ax + a^2$ by $x + a$. *Ans.* $x^3 + 3ax^2 + 3a^2x + a^3$.

3. $x^3 + y^3$ by $x + y$. *Ans.* $x^4 + xy^3 + x^3y + y^4$.

4. $3ab^2 + 6a^2c^2$ by $3ab^2 + 3a^2c^2$.
Ans. $9a^2b^4 + 27a^3b^2c^2 + 18a^4c^4$.

5. $a^2b^2 + c^2d$ by $a + b$. *Ans.* $a^3b^2 + ac^2d + a^2b^3 + bc^2d$.

6. $3ax^2 + 9ab^3 + cd^5$ by $6a^2c^2$.
Ans. $18a^3c^2x^2 + 54a^3c^2b^3 + 6a^2c^3d^5$.

7. $64a^3x^3 + 27a^2x + 9ab$ by $8a^3cd$.
Ans. $512a^6cdx^3 + 216a^5cdx + 72a^4bcd$.

8. $a^3 + 3a^2x + 3ax^2 + x^3$ by $a + x$.
Ans. $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$.

9. $x^2 + y^2$ by $x + y$. *Ans.* $x^3 + xy^2 + x^2y + y^3$.

10. $x^5 + xy^6 + 7ax$ by $ax + 5ax$.
Ans. $6ax^6 + 6ax^2y^6 + 42a^2x^2$.

11. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.
Ans. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

12. $x^3 + x^2y + xy^2 + y^3$ by $x + y$.
Ans. $x^4 + 2x^3y + 2x^2y^2 + 2xy^3 + y^4$.

13. $x^3 + 2x^2 + x + 3$ by $3x + 1$.
Ans. $3x^4 + 7x^3 + 5x^2 + 10x + 3$.

FOR REVIEW.

Multiply the following: —

1. $2ax - 3ab$ by $3x - b$.

$$\begin{array}{l} 2ax - 3ab \\ \underline{3x \;\; - \;\; b \quad} \\ 6ax^2 - 9abx \\ \underline{\qquad - 2abx + 3ab^2} \\ 6ax^2 - 11abx + 3ab^2 \end{array}$$

2. $a^4 - 2b^3$ by $a - b$. *Ans.* $a^5 - 2ab^3 - a^4b + 2b^4$.

3. $x^2 - 3x - 7$ by $x - 2$. *Ans.* $x^3 - 5x^2 - x + 14$.

4. $3a^2 - 5ab + 2b^2$ by $a^2 - 7ab$.
Ans. $3a^4 - 26a^3b + 37a^2b^2 - 14ab^3$.

5. $b^2 + b^4 + b^6$ by $b^2 - 1$. *Ans.* $b^8 - b^2$.

6. $x^4 - 2x^3y + 4x^2y^2 - 8xy^3 + 16y^4$ by $x + 2y$.
Ans. $x^5 + 32y^5$.

7. $4x^2 - 2y$ by $2y$. *Ans.* $8x^2y - 4y^2$.

8. $2x + 4y$ by $2x - 4y$. *Ans.* $4x^2 - 16y^2$.

9. $x^3 + x^2y + xy^2 + y^3$ by $x - y$. *Ans.* $x^4 - y^4$.

10. $x^2 + xy + y^2$ by $x^2 - xy + y^2$. *Ans.* $x^4 + x^2y^2 + y^4$.

11. $2a^2 - 3ax + 4x^2$ by $5a^2 - 6ax - 2x^2$.
Ans. $10a^4 - 27a^3x + 34a^2x^2 - 18ax^3 - 8x^4$.

12. $3x^2 - 2xy + 5$ by $x^2 + 2xy - 3$.
Ans. $3x^4 + 4x^3y - 4x^2 - 4x^2y^2 + 16xy - 15$.

13. $3x^3 + 2x^2y^2 + 3y^2$ by $2x^3 - 3x^2y^2 + 5y^3$.
Ans. $6x^6 - 5x^5y^2 - 6x^4y^4 + 6x^3y^2 + 15x^3y^3 - 9x^2y^4$
$+ 10x^2y^5 + 15y^5$.

14. $8ax - 6ab - c$ by $2ax + ab + c$.
Ans. $16a^2x^2 - 4a^2bx - 6a^2b^2 + 6acx - 7abc - c^2$.

15. $3a^2 - 5b^2 + 3c^2$ by $a^2 - b^2$.
Ans. $3a^4 - 8a^2b^2 + 3a^2c^2 + 5b^4 - 3b^2c^2$.

16.
$$\begin{array}{r} 3a^2 - 5bd + cf \\ -\ 5a^2 + 4bd - 8cf \\ \hline -15a^4 + 37a^2bd - 29a^2cf - 20b^2d^2 + 44bcdf - 8c^2f^2 \end{array}$$

17. $a^mx - a^2b^2$ by a^2x^n. *Ans.* $a^{m+2}x^{n+1} - a^4b^2x^n$.

18. $a^m + b^n$ by $a^m - b^n$. *Ans.* $a^{2m} - b^{2n}$.

19. $a^m + b^n$ by $a^m + b^n$. *Ans.* $a^{2m} + 2a^mb^n + b^{2n}$.

Division.

45. Division is the operation of finding from two quantities a third, which, being multiplied by the second, will produce the first.

The first is called the **dividend**; the second, the **divisor**; and the third, the **quotient**.

Division is the converse of multiplication. In it we have given the product and one factor to find the other. The rules for division are just the converse of those for multiplication.

46. To Divide One Monomial by Another.

Divide $72a^5$ by $8a^3$.

The division is indicated thus: $\frac{72a^5}{8a^3}$.

The quotient must be such a monomial as, being *multiplied by the divisor*, will give the dividend: hence the coefficient of the quotient must be 9, and the literal part a^2; for these quantities multiplied by $8a^3$ will give $72a^5$. Hence

$$\frac{72a^5}{8a^3} = 9a^2.$$

The coefficient 9 is obtained by dividing 72 by 8; and the literal part is found by giving to a an exponent equal to 5 minus 3.

For dividing one monomial by another, then, we deduce from the above example the following rule: —

Divide the coefficient of the dividend by the coefficient of the divisor, for a new coefficient.

After this coefficient write all the letters of the dividend, giving to each an exponent equal to the excess of its exponent in the dividend over that in the divisor.

47. Signs in Division. Since the quotient multiplied by the divisor must produce the dividend, and since the product of two factors having the *same sign* will be +, and the product of two factors having different signs will be −, we conclude, —

(1) *When the signs of the dividend and divisor are like, the sign of the quotient will be* +.

(2) *When the signs of the dividend and divisor are unlike, the sign of the quotient will be* −.

Again, for brevity we say,—

+ divided by +, and − divided by −, give +.

− divided by +, and + divided by −, give −.

$$\frac{+ab}{+a} = +b; \qquad \frac{-ab}{-b} = +a;$$

$$\frac{-ab}{+a} = -b; \qquad \frac{+ab}{-a} = -b.$$

Exercises.

1. $\dfrac{+18a^5b^2c}{+9a^3bc} = +2a^2b.$

2. $\dfrac{-15a^3x^2y}{-5a^2x} = +3axy.$

3. $\dfrac{-24a^4bc}{+3abc} = -8a^3.$

4. $\dfrac{32a^8b^3x^5}{-8a^7b^3x} = -4ax^4.$

Divide the following:—

5. $15ax^2y^3$ by $-3ay$. *Ans.* $-5x^2y^2$.
6. $84ab^3x$ by $12b^2$. *Ans.* $7abx$.
7. $-36a^4b^5c^2$ by $9a^3b^2c$. *Ans.* $-4ab^3c$.
8. $-99a^4b^4x^5$ by $11a^3b^2x^4$. *Ans.* $-9ab^2x$.
9. $108x^6y^5z^3$ by $54x^5z$. *Ans.* $2xy^5z^2$.
10. $64x^7y^5z^6$ by $-16x^6y^4z^5$. *Ans.* $-4xyz$.
11. $-96a^7b^6c^5$ by $12a^2bc$. *Ans.* $-8a^5b^5c^4$.
12. $-38a^4b^6d^4$ by $2a^3b^5d$. *Ans.* $-19abd^3$.
13. $-64a^5b^4c^8$ by $32a^4bc$. *Ans.* $-2ab^3c^7$.
14. $128a^5x^6y^7$ by $16axy^4$. *Ans.* $8a^4x^5y^3$.
15. $-256a^4b^9c^8d^7$ by $16a^3bc^6$. *Ans.* $-16ab^8c^2d^7$.
16. $200a^8m^2n^2$ by $-50a^7mn$. *Ans.* $-4amn$.
17. $300x^3y^4z^2$ by $60xy^2z$. *Ans.* $5x^2y^2z$.
18. $27a^5b^2c^2$ by $-9abc$. *Ans.* $-3a^4bc$.
19. $64a^3y^6z^8$ by $32ay^5z^7$. *Ans.* $2a^2yz$.

20. $-88a^5b^6c^8$ by $11a^3b^4c^6$. *Ans.* $-8a^2b^2c^2$.

21. $77a^4y^3z^4$ by $-11a^4y^3z^4$. *Ans.* -7.

22. $84a^4b^2c^2d$ by $-42a^4b^2c^2d$. *Ans.* -2.

23. $-88a^6b^7c^6$ by $8a^5b^6c^6$. *Ans.* $-11ab$.

24. $16x^2$ by $-8x$. *Ans.* $-2x$.

25. $-88a^nb^2$ by $11a^mb$. *Ans.* $-8a^{n-m}b$.

26. $77a^mb^n$ by $-11a^2b^3$. *Ans.* $-7a^{m-2}b^{n-3}$.

27. $84a^8b^m$ by $42a^nb^9$. *Ans.* $2a^{8-n}b^{m-9}$.

28. $-88a^pb^q$ by $8a^nb^m$. *Ans.* $-11a^{p-n}b^{q-m}$.

29. $96ab^p$ by $48a^nb^q$. *Ans.* $2a^{1-n}b^{p-q}$.

30. $168x^ay^b$ by $12x^ny^m$. *Ans.* $14x^{a-n}y^{b-m}$.

31. $256ab^3c^2$ by $16a^nb^mc^p$. *Ans.* $16a^{1-n}b^{3-m}c^{2-p}$.

48. It follows from the preceding rules that the exact division of monomials will be impossible

(1) *When the coefficient of the dividend is not exactly divisible by that of the divisor.*

(2) *When the exponent of the same letter is greater in the divisor than in the dividend.*

(3) *When the divisor contains one or more letters not found in the dividend.*

In any of the above cases the quotient will be expressed by a fraction.

A fraction is said to be in its *simplest form* when the numerator and the denominator do not contain a common factor: for example, $12a^4b^2cd$ divided by $8a^2bc^2$ gives $\frac{12a^4b^2cd}{8a^2bc^2}$, which may be reduced by dividing the numerator and denominator by the common factors, 4, a^2, b, and c, giving

$$\frac{12a^4b^2cd}{8a^2bc^2}=\frac{3a^2bd}{2c};\ \text{also}\ \frac{25a^5b^2d^3}{15a^4b^6d^4}=\frac{5a}{3b^4d}.$$

Hence, for the reduction of a monomial fraction to its simplest form, we have the following rule: —

Suppress every factor, whether numerical or literal, that is common to both terms of the fraction. The result will be the reduced fraction sought.

Exercises.

1. $\dfrac{48a^3b^2cd^3}{36a^2b^3c^2de}=\dfrac{4ad^2}{3bce}$.

2. $\dfrac{37ab^3c^5d}{6a^3bc^4d^2}=\dfrac{37b^2c}{6a^2d}$.

3. $\dfrac{7a^2b}{14a^3b^2}=\dfrac{1}{2ab}$.

4. $\dfrac{4a^2b^2}{6ab^4}=\dfrac{2a}{3b^2}$.

Divide

5. $49a^2b^2c^6$ by $14a^3bc^4$. *Ans.* $\dfrac{7bc^2}{2a}$.

6. $6amn$ by $3abc$. *Ans.* $\dfrac{2mn}{bc}$.

7. $18a^2b^2mn^2$ by $12a^4b^4cd$. *Ans.* $\dfrac{3mn^2}{2a^2b^2cd}$.

8. $28a^5b^6c^7d^8$ by $16ab^9cd^7m$. *Ans.* $\dfrac{7a^4c^6d}{4b^3m}$.

9. $72a^3c^2b^2$ by $12a^5c^4b^3d$. *Ans.* $\dfrac{6}{a^2c^2bd}$.

10. $100a^8b^5xmn$ by $25a^3b^4d$. *Ans.* $\dfrac{4a^5bxmn}{d}$.

11. $96a^5b^8c^9df$ by $75a^2cxy$. *Ans.* $\dfrac{32a^3b^8c^8df}{25xy}$.

12. $85m^2n^3fx^2y^3$ by $15am^4nf$. *Ans.* $\dfrac{17n^2x^2y^3}{3am^2}$.

13. $127d^3x^2y^2$ by $16d^4x^4y^4$. *Ans.* $\dfrac{127}{16dx^2y^2}$.

49. In dividing monomials it often happens that the exponents of the same letter in the dividend and divisor are equal, in which case that letter may not appear in the quotient. It might, however, be retained by giving to it the exponent 0.

If we have expressions of the form

$$\frac{a}{a},\ \frac{a^2}{a^2},\ \frac{a^3}{a^3},\ \frac{a^4}{a^4},\ \frac{a^5}{a^5},\ \text{etc.,}$$

and apply the rule for the exponents, we shall have

$$\frac{a}{a}=a^{1-1}=a^0,\ \frac{a^2}{a^2}=a^{2-2}=a^0,\ \frac{a^3}{a^3}=a^{3-3}=a^0,\ \text{etc.}$$

But since any quantity divided by itself is equal to 1, it follows that

$$\frac{a}{a}=a^0=1,\ \frac{a^2}{a^2}=a^{2-2}=a^0=1,\ \text{etc.};$$

or, finally, if we designate the exponent by m, we have

$$\frac{a^m}{a^m}=a^{m-m}=a^0=1;\ \text{that is,}$$

The 0 power of any quantity is equal to 1: therefore

Any quantity may be retained in a term, or introduced into a term, by giving it the exponent 0.

Exercises.

1. Divide $6a^2b^2c^4$ by $2a^2b^2$.

$$\frac{6a^2b^2c^4}{2a^2b^2}=3a^{2-2}b^{2-2}c^4=3a^0b^0c^4=3c^4.$$

2. Divide $8a^4b^3c^5$ by $-4a^4b^3c$. *Ans.* $-2a^0b^0c^4=-2c^4$.

3. Divide $-32m^3n^2x^2y^2$ by $4m^3n^2xy$.

Ans. $-8m^0n^0xy=-8xy$.

4. Divide $-96a^4b^5c^n$ by $-24a^4b^5$. *Ans.* $4a^0b^0c^n = 4c^n$.

5. Introduce a as a factor into $6b^5c^4$. *Ans.* $6a^0b^5c^4$.

6. Introduce ab as factors into $9c^5d^n$. *Ans.* $9a^0b^0c^5d^n$.

7. Introduce abc as factors into $8d^4f^m$. *Ans.* $8a^0b^0c^0d^4f^m$.

50. When the exponent of any letter is greater in the divisor than it is in the dividend, the exponent of that letter in the quotient may be written with a negative sign. Thus,

$$\frac{a^2}{a^5} = \frac{1}{a^3}; \text{ also } \frac{a^2}{a^5} = a^{2-5} = a^{-3}, \text{ by the rule:}$$

hence $$a^{-3} = \frac{1}{a^3}.$$

Since $$a^{-3} = \frac{1}{a^3}, \text{ we have } b \times a^{-3} = \frac{b}{a^3};$$

that is, a in the numerator with a negative exponent is equal to a in the denominator with an equal positive exponent. Hence

Any quantity having a negative exponent is equal to the reciprocal of the same quantity with an equal positive exponent.

Any factor may be transferred from the denominator to the numerator of a fraction, or the reverse, by changing the sign of its exponent.

Exercises.

1. Divide $32a^2bc$ by $16a^5b^2$.

Ans. $\dfrac{32a^2bc}{16a^5b^2} = 2a^{-3}b^{-1}c = \dfrac{2c}{a^3b}$.

2. $\dfrac{54a^2b^3c}{9a^4b^5} = 6a^{-2}b^{-2}c$. *Ans.* $\dfrac{6c}{a^2b^2}$.

3. Reduce $\dfrac{17x^2y^3z}{51x^4y^3}$. *Ans.* $\dfrac{x^{-2}z}{3}$, or $\dfrac{z}{3x^2}$.

4. In $5ay^{-3}x^{-2}$ get rid of the negative exponents.

Ans. $\dfrac{5a}{y^3x^2}$.

5. In $\dfrac{4a^2b^3x^{-2}}{3a^{-3}b^{-5}}$ get rid of the negative exponents.

Ans. $\dfrac{4a^5b^8}{3x^2}$.

6. In $\dfrac{15a^{-3}c^{-4}d^{-5}}{45x^{-3}y^{-5}c^{-2}}$ get rid of the negative exponents.

Ans. $\dfrac{x^3y^5}{3a^3c^2d^5}$.

7. Reduce $\dfrac{-8a^{-3}b^5c}{14a^2b^{-5}c}$. *Ans.* $\dfrac{-4a^{-5}b^{10}c^0}{7}$, or $\dfrac{-4b^{10}}{7a^5}$.

8. Reduce $72a^5b^2 \div 8a^6b^3$. *Ans.* $9a^{-1}b^{-1}$, or $\dfrac{9}{ab}$.

9. In $\dfrac{15a^{-4}b^6c^{-1}}{5a^{-2}b^{-1}}$ get rid of the negative exponents.

Ans. $\dfrac{3b^7}{a^2c}$.

10. Reduce $\dfrac{-15a^{-5}b^{-5}c^2}{-5a^{-6}b^{-7}}$. *Ans.* $3ab^2c^2$.

51. To Divide a Polynomial by a Monomial.

Divide each term of the dividend separately by the divisor. The algebraic sum of the quotients will be the quotient sought.

Exercises.

Divide

1. $3a^2b^2 - a$ by a. *Ans.* $3ab^2 - 1$.
2. $5a^3b^2 - 25a^4b^2$ by $5a^3b^2$. *Ans.* $1 - 5a$.
3. $35a^2b^2 - 25ab$ by $-5ab$. *Ans.* $-7ab + 5$.
4. $10ab - 15ac$ by $5a$. *Ans.* $2b - 3c$.
5. $6ab - 8ax + 4a^2y$ by $2a$. *Ans.* $3b - 4x + 2ay$.

6. $-15ax^2+6x^3$ by $-3x$. *Ans.* $5ax-2x^2$.

7. $-21xy^2+35a^2b^2y-7c^2y$ by $-7y$. *Ans.* $3xy-5a^2b^2+c^2$.

8. $40a^8b^4+8a^4b^7-32a^4b^4c^4$ by $8a^4b^4$. *Ans.* $5a^4+b^3-4c^4$.

52. Division of Polynomials.

(1) Divide $-2a+6a^2-8$ by $2+2a$.

$$\begin{array}{lll}
\textit{Dividend.} & \textit{Divisor.} & \\
6a^2-2a-8 & \underline{|\,2a+2} & \\
\underline{6a^2+6a} & 3a-4 & \textit{Quotient.} \\
\quad -8a-8 & & \\
\quad \underline{-8a-8} & & \\
\qquad 0 & \textit{Remainder.} &
\end{array}$$

We first arrange the dividend and divisor with reference to a (§ 44), placing the divisor on the right of the dividend. Divide the first term of the dividend by the first term of the divisor. The result will be the first term of the quotient, which, for convenience, we place under the divisor. The product of the divisor by this term ($6a^2+6a$), being subtracted from the dividend, leaves a new dividend, which may be treated in the same way as the original one; and so on to the end of the operation.

Since all similar cases may be treated in the same way, we have, for the division of polynomials, the following rule: —

Arrange the dividend and the divisor with reference to the same letter.

Divide the first term of the dividend by the first term of the divisor, for the first term of the quotient. Multiply the divisor by this term of the quotient, and subtract the product from the dividend.

Divide the first term of the remainder by the first term of the divisor, for the second term of the quotient. Multiply the divisor by this term, and subtract the product from the first remainder, and so on.

Continue the operation until a remainder is found equal to 0, *or one whose first term is not divisible by that of the divisor.*

NOTES. — 1. When a remainder is found equal to 0, the division is exact.

2. When a remainder is found whose first term is not divisible by the first term of the divisor, the exact division is impossible. In that case, write the last remainder after the quotient found, placing the divisor under it, in the form of a fraction.

(2) Let it be required to divide

$51a^2b^2 + 10a^4 - 48a^3b - 15b^4 + 4ab^3$ by $4ab - 5a^2 + 3b^2$.

NOTE. — First arrange the dividend and divisor with reference to a.

$$\begin{array}{l|l}
\textit{Dividend.} & \textit{Divisor.} \\
10a^4 - 48a^3b + 51a^2b^2 + 4ab^3 - 15b^4 & -5a^2 + 4ab + 3b^2 \\
\underline{+10a^4 - 8a^3b - 6a^2b^2} & -2a^2 + 8ab - 5b^2 \\
\quad -40a^3b + 57a^2b^2 + 4ab^3 - 15b^4 & \textit{Quotient.} \\
\quad \underline{-40a^3b + 32a^2b^2 + 24ab^3} & \\
\qquad\qquad 25a^2b^2 - 20ab^3 - 15b^4 & \\
\qquad\qquad \underline{25a^2b^2 - 20ab^3 - 15b^4} &
\end{array}$$

(3)
$$\begin{array}{l|l}
x^4 + x^3y + x^2y + xy^2 - 2y & x + y \\
\underline{x^4 + x^3y} & x^3 + xy - \dfrac{2y}{x+y}. \\
\qquad + x^2y + xy^2 & \\
\qquad \underline{+ x^2y + xy^2} & \\
\qquad\qquad -2y &
\end{array}$$

NOTE. — Here the division is not exact, and the quotient is fractional.

(4)
$$\begin{array}{l|l}
1 + a & 1 - a \\
\underline{1 - a} & 1 + 2a + 2a^2 + 2a^3 + \text{, etc.} \\
+2a & \\
\underline{+2a - 2a^2} & \\
\quad +2a^2 & \\
\quad \underline{+2a^2 - 2a^3} & \\
\qquad +2a^3 &
\end{array}$$

NOTE. — In this example the operation does not terminate: it may be continued to any extent.

Exercises.

Divide

1. $a^2+2ax+x^2$ by $a+x$. *Ans.* $a+x$.

2. $a^3-3a^2y+3ay^2-y^3$ by $a-y$. *Ans.* $a^2-2ay+y^2$.

3. $24a^2b-12a^3cb^2-6ab$ by $-6ab$.
Ans. $-4a+2a^2cb+1$.

4. $6x^4-96$ by $3x-6$. *Ans.* $2x^3+4x^2+8x+16$.

5. $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$.
Ans. $a^3-3a^2x+3ax^2-x^3$.

6. $48x^3-76ax^2-64a^2x+105a^3$ by $2x-3a$.
Ans. $24x^2-2ax-35a^2$.

7. $y^6-3y^4x^2+3y^2x^4-x^6$ by $y^3-3y^2x+3yx^2-x^3$.
Ans. $y^3+3y^2x+3yx^2+x^3$.

8. $64a^4b^6-25a^2b^8$ by $8a^2b^3+5ab^4$. *Ans.* $8a^2b^3-5ab^4$.

9. $6a^3+23a^2b+22ab^2+5b^3$ by $3a^2+4ab+b^2$.
Ans. $2a+5b$.

10. $6ax^6+6ax^2y^6+42a^2x^2$ by $ax+5ax$.
Ans. x^5+xy^6+7ax.

11. $-15a^4+37a^2bd-29a^2cf-20b^2d^2+44bcdf-8c^2f^2$ by $3a^2-5bd+cf$. *Ans.* $-5a^2+4bd-8cf$.

12. $x^4+x^2y^2+y^4$ by x^2-xy+y^2. *Ans.* x^2+xy+y^2.

13. x^4-y^4 by $x-y$. *Ans.* $x^3+x^2y+xy^2+y^3$.

14. $3a^4-8a^2b^2+3a^2c^2+5b^4-3b^2c^2$ by a^2-b^2.
Ans. $3a^2-5b^2+3c^2$.

15. $6x^6-5x^5y^2-6x^4y^4+6x^3y^2+15x^3y^3-9x^2y^4+10x^2y^5+15y^5$ by $3x^3+2x^2y^2+3y^2$.
Ans. $2x^3-3x^2y^2+5y^3$.

16. $-c^2+16a^2x^2-7abc-4a^2bx-6a^2b^2+6acx$ by $8ax-6ab-c$. *Ans.* $2ax+ab+c$.

17. $3x^4+4x^3y-4x^2-4x^2y^2+16xy-15$ by $2xy+x^2-3$. *Ans.* $3x^2-2xy+5$.

18. x^5+32y^5 by $x+2y$.
Ans. $x^4-2x^3y+4x^2y^2-8xy^3+16y^4$.

19. $3a^4-26a^3b-14ab^3+37a^2b^2$ by $2b^2-5ab+3a^2$.
Ans. a^2-7ab.

20. a^4-b^4 by $a^3+a^2b+ab^2+b^3$. *Ans.* $a-b$.

21. $x^3-3x^2y+y^3$ by $x+y$. *Ans.* $x^2-4xy+4y^2-\dfrac{3y^3}{x+y}$.

22. $1+2a$ by $1-a-a^2$. *Ans.* $1+3a+4a^2+7a^3+$, etc.

CHAPTER III.

FACTORING, GREATEST COMMON DIVISOR, AND LEAST COMMON MULTIPLE.

USEFUL FORMULAS.

53. A **formula** is an algebraic expression of a general rule or principle.

Formulas serve to shorten algebraic operations, and are also of much use in the operation of factoring. When translated into common language, they give rise to practical rules.

The verification of the following formulas affords additional exercises in multiplication and division.

54. Formula 1. — To form the square of $a + b$, we have

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2 = a^2 + b^2 + 2ab;$$

that is,

The square of the sum of any two quantities is equal to the sum of their squares, plus twice their product.

(1) Find the square of $2a + 3b$.

We have from the rule,

$$(2a+3b)^2 = 4a^2 + 9b^2 + 12ab.$$

(2) Find the square of $5ab + 3ac$.

Ans. $25a^2b^2 + 9a^2c^2 + 30a^2bc$.

(3) Find the square of $5a^2 + 8a^2b$.

Ans. $25a^4 + 64a^4b^2 + 80a^4b$.

(4) Find the square of $6ax + 9a^2x^2$.

Ans. $36a^2x^2 + 81a^4x^4 + 108a^3x^3$.

NOTE. — If the expression to be squared consists of more than two terms, it may be written in the *form* of a binomial, and squared. Thus,

$$(a+b+c)^2=[a+(b+c)]^2=a^2+(b+c)^2+2a(b+c)$$
$$=a^2+b^2+c^2+2ab+2ac+2bc;$$

$$(a+b+c+d)^2=[(a+b)+(c+d)]^2$$
$$=(a+b)^2+(c+d)^2+2(a+b)(c+d)$$
$$=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

Find the square of the following: —

(5) $2a+3b+c$. (6) $2x+3y+4b$. (7) $x+a+2y$.

55. Formula 2. — To form the square of a difference, $a-b$, we have

$$(a-b)^2=(a-b)(a-b)=a^2-2ab+b^2=a^2+b^2-2ab;$$

that is,

The square of the difference of any two quantities is equal to the sum of their squares, minus twice their product.

(1) Find the square of $2a-b$.

We have $(2a-b)^2=4a^2+b^2-4ab$.

(2) Find the square of $4ac-bc$.

Ans. $16a^2c^2+b^2c^2-8abc^2$.

(3) Find the square of $7a^2b^2-12ab^3$.

Ans. $49a^4b^4+144a^2b^6-168a^3b^5$.

NOTE. — Similarly,

$$(a-b+c)^2=[a-(b-c)]^2=a^2+(b-c)^2-2a(b-c)$$
$$=a^2+b^2+c^2-2ab+2ac-2bc;$$

and

$$(a-b-c-d)^2=[(a-b)-(c+d)]^2$$
$$=(a-b)^2+(c-d)^2-2(a-b)(c+d)$$
$$=a^2+b^2+c^2+d^2-2ab-2ac-2ad+2bc+2bd+2cd.$$

56. Formula 3. — To multiply $a+b$ by $a-b$, we have

$$(a+b)\times(a-b)=a^2-b^2;$$

that is,

The sum of two quantities, multiplied by their difference, is equal to the difference of their squares.

(1) Multiply $2c+b$ by $2c-b$. *Ans.* $4c^2-b^2$.

(2) Multiply $9ac+3bc$ by $9ac-3bc$.
Ans. $81a^2c^2-9b^2c^2$.

(3) Multiply $8a^3+7ab^2$ by $8a^3-7ab^2$.
Ans. $64a^6-49a^2b^4$.

NOTE. — To multiply $a+b-c$ by $a-b+c$, we have

$$(a+b-c)\times(a-b+c)=[a+(b-c)]\times[a-(b-c)]=a^2-(b-c)^2.$$

Multiply together the following: —

(4) $x+y-4$ and $x-y+4$.

(5) $2a-b-d$ and $2a+b+d$.

57. Formula 4. — To multiply a^2+ab+b^2 by $a-b$, we have

$$(a^2+ab+b^2)(a-b)=a^3-b^3.$$

58. Formula 5. — To multiply a^2-ab+b^2 by $a+b$, we have

$$(a^2-ab+b^2)(a+b)=a^3+b^3.$$

59. Formula 6. — To multiply together $a+b$, $a-b$, and a^2+b^2, we have

$$(a+b)(a-b)(a^2+b^2)=a^4-b^4.$$

60. Since every product is divisible by any of its factors, each formula establishes the principle set opposite its number.

(1) *The sum of the squares of any two quantities, plus twice their product, is divisible by their sum.*

(2) *The sum of the squares of any two quantities, minus twice their product, is divisible by the difference of the quantities.*

(3) *The difference of the squares of any two quantities is divisible by the sum of the quantities, and also by their difference.*

(4) *The difference of the cubes of any two quantities is divisible by the difference of the quantities; also by the sum of their squares, plus their product.*

(5) *The sum of the cubes of any two quantities is divisible by the sum of the quantities; also by the sum of their squares minus their product.*

(6) *The difference between the fourth powers of any two quantities is divisible by the sum of the quantities, by their difference, by the sum of their squares, and by the difference of their squares.*

FACTORING.

61. Factoring is the operation of resolving a quantity into factors. The principles employed are the converse of those of multiplication. The operations of factoring are performed by inspection. Thus,

(1) What are the factors of the polynomial $ac + ab + ad$?

We see by inspection that a is a common factor of all the terms: hence it may be placed without a parenthesis, and the other parts within. Thus, $ac + ab + ad = a(c + b + d)$.

(2) Find the factors of the polynomial $a^2b^2 + a^2d - a^2f$.
Ans. $a^2(b^2 + d - f)$.

(3) Find the factors of the polynomial $3a^2b - 6a^2b^2 + b^2d$.
Ans. $b(3a^2 - 6a^2b + bd)$.

(4) Find the factors of $3a^2b - 9a^2c - 18a^2xy$.
Ans. $3a^2(b - 3c - 6xy)$.

(5) Find the factors of $8a^2cx - 18acx^2 + 2ac^5y - 30a^6c^9$.
Ans. $2ac(4ax - 9x^2 + c^4y - 15a^5c^8)$.

(6) Factor $30a^4b^2c - 6a^3b^2d^3 + 18a^3b^2c^2$.
Ans. $6a^3b^2(5ac - d^3 + 3c^2)$.

(7) Factor $12c^4bd^3 - 15c^3d^4 - 6c^3d^3f$.
Ans. $3c^3d^3(4c^2b - 5cd - 2f)$.

(8) Factor $15a^3bcf - 10abc^4 - 25abcd$.
Ans. $5abc(3a^2f - 2c^3 - 5d)$.

62. When two terms of a trinomial are squares, and positive, and the third term is equal to twice the product of their square roots, the trinomial may be resolved into factors by Formula 1. Thus,

Factor the following: —

(1) $a^2 + 2ab + b^2$. *Ans.* $(a + b)(a + b)$.

(2) $4a^2 + 12ab + 9b^2$. *Ans.* $(2a + 3b)(2a + 3b)$.

(3) $9a^2 + 12ab + 4b^2$. *Ans.* $(3a + 2b)(3a + 2b)$.

(4) $4x^2 + 8x + 4$. *Ans.* $(2x + 2)(2x + 2)$.

(5) $9a^2b^2 + 12abc + 4c^2$. *Ans.* $(3ab + 2c)(3ab + 2c)$.

(6) $16x^2y^2 + 16xy^3 + 4y^4$. *Ans.* $(4xy + 2y^2)(4xy + 2y^2)$.

NOTE. — Factor the following by note to Formula 1.

(7) $a^2 + 2ab + 2ac + b^2 + c^2 + 2bc = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$. *Ans.* $(a + b + c)(a + b + c)$.

(8) $x^2 + 4a^2 + 9b^2 + 4ax + 6bx + 12ab$.
Ans. $(x + 2a + 3b)(x + 2a + 3b)$.

63. When two terms of a trinomial are squares, and positive, and the third term is equal to *minus* twice their square roots, the trinomial may be factored by Formula 2. Thus,

Factor the following: —

(1) $a^2 - 2ab + b^2$. *Ans.* $(a-b)(a-b)$.

(2) $4a^2 - 4ab + b^2$. *Ans.* $(2a-b)(2a-b)$.

(3) $9a^2 + c^2 - 6ac$. *Ans.* $(3a-c)(3a-c)$.

(4) $a^2x^2 - 4ax + 4$. *Ans.* $(ax-2)(ax-2)$.

(5) $4x^2 + y^2 - 4xy$. *Ans.* $(2x-y)(2x-y)$.

(6) $36x^2 - 24xy + 4y^2$. *Ans.* $(6x-2y)(6x-2y)$.

NOTE. — Factor the following by note to Formula 2.

(7) $a^2 - 2ab + b^2 - 2ac + c^2 + 2bc = a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$.

Ans. $(a-b-c)(a-b-c) = [a-(b+c)][a-(b+c)]$.

(8) $4x^2 - 4xy + 12bx + y^2 + 9b^2 - 6by$.

Ans. $(2x-y+3b)(2x-y+3b) = [2x-(y-3b)][2x-(y-3b)]$.

64. When the two terms of a binomial are squares, and have contrary signs, the binomial may be factored by Formula 3. Thus,

Factor the following: —

(1) $4c^2 - b^2$. *Ans.* $(2c+b)(2c-b)$.

(2) $81a^2c^2 - 9b^2c^2$. *Ans.* $(9ac+3bc)(9ac-3bc)$.

(3) $64a^4b^4 - 25x^2y^2$. *Ans.* $(8a^2b^2+5xy)(8a^2b^2-5xy)$.

(4) $25a^2c^2 - 9x^4y^2$. *Ans.* $(5ac+3x^2y)(5ac-3x^2y)$.

(5) $36a^4b^4c^2 - 9x^6$. *Ans.* $(6a^2b^2c+3x^3)(6a^2b^2c-3x^3)$.

(6) $49x^4 - 36y^4$. *Ans.* $(7x^2+6y^2)(7x^2-6y^2)$.

NOTE. — Factor the following by note to Formula 3.

(7) $4a^2 - 16(b-c)^2$. *Ans.* $[2a-4(b-c)][2a+4(b-c)]$.

(8) $(x-3y)^2 - 16m^2$. *Ans.* $(x-3y+4m)(x-3y-4m)$.

65. When the two terms of a binomial are cubes, and have contrary signs, the binomial may be factored by Formula 4. Thus,

Factor the following: —

(1) $8a^3 - c^3$. *Ans.* $(2a - c)(4a^2 + 2ac + c^2)$.

(2) $27a^3 - 64$. *Ans.* $(3a - 4)(9a^2 + 12a + 16)$.

(3) $a^3 - 64b^3$. *Ans.* $(a - 4b)(a^2 + 4ab + 16b^2)$.

(4) $a^3 - 27b^3$. *Ans.* $(a - 3b)(a^2 + 3ab + 9b^2)$.

66. When the terms of a binomial are cubes, and have like signs, the binomial may be factored by Formula 5. Thus,

Factor the following: —

(1) $8a^3 + c^3$. *Ans.* $(2a + c)(4a^2 - 2ac + c^2)$.

(2) $27a^3 + 64$. *Ans.* $(3a + 4)(9a^2 - 12a + 16)$.

(3) $a^3 + 64b^3$. *Ans.* $(a + 4b)(a^2 - 4ab + 16b^2)$.

(4) $a^3 + 27b^3$. *Ans.* $(a + 3b)(a^2 - 3ab + 9b^2)$.

67. When the terms of a binomial are 4th powers, and have contrary signs, the binomial may be factored by Formula 6. Thus,

What are the factors of

(1) $a^4 - b^4$? *Ans.* $(a + b)(a - b)(a^2 + b^2)$.

(2) $81a^4 - 16b^4$? *Ans.* $(3a + 2b)(3a - 2b)(9a^2 + 4b^2)$.

(3) $16a^4b^4 - 81c^4d^4$?
Ans. $(2ab + 3cd)(2ab - 3cd)(4a^2b^2 + 9c^2d^2)$.

(4) $8a^4x^4 - 625c^4y^4$?
Ans. $(2ax + 5cy)(2ax - 5cy)(4a^2x^2 + 25c^2y^2)$.

67a. Verify this formula by multiplication: —

$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

By it the following expressions may be factored: —

(1) $x^2+17x+70$. *Ans.* $(x+7)(x+10)$.

(2) $x^2-18x+80$. *Ans.* $(x-8)(x-10)$.

(3) $x^2+5x-36$. *Ans.* $(x+9)(x-4)$.

(4) $x^2-8x-33$. *Ans.* $(x+3)(x-11)$.

GREATEST COMMON DIVISOR.

68. A **common divisor** of two quantities is a quantity that will divide each of them without a remainder. Thus, $3a^2b$ is a common divisor of $9a^2b^2c$ and $3a^2b^2-6a^3b^3$.

69. A **simple** or **prime factor** is one that cannot be resolved into any other factors.

Every prime factor common to two quantities is a common divisor of those quantities. The continued product of any number of prime factors common to two quantities is also a common divisor of those quantities.

70. The **greatest common divisor** of two quantities is the continued product of all the prime factors which are common to both.

71. When both quantities can be resolved into prime factors by the method of factoring already given, the greatest common divisor may be found by the following rule: —

Resolve both quantities into their prime factors.

Find the continued product of all the factors which are common to both, and it will be the greatest common divisor required.

Exercises.

NOTE. — For convenience, G.C.D. will be used to denote greatest common divisor.

Find the G. C. D. of

1. $75a^2b^2c$ and $25abd$.

Factoring, we have

$$75a^2b^2c = 3 \times 5 \times 5aabbc,$$
$$25abd = 5 \times 5abd.$$

The factors, 5, 5, a, and b are common: hence

$$5 \times 5 \times a \times b = 25ab = \text{the divisor sought.}$$

VERIFICATION. $75a^2b^2c \div 25ab = 3abc,$
$25abd \div 25ab = d;$

and, since the quotients have no common factor, they cannot be further divided.

2. $a^2 - 2ab + b^2$ and $a^2 - b^2$. *Ans.* $a - b$.
3. $a^2 + 2ab + b^2$ and $a + b$. *Ans.* $a + b$.
4. $a^2x^2 - 4ax + 4$ and $ax - 2$. *Ans.* $ax - 2$.
5. $3a^2b - 9a^2c - 18a^2xy$ and $b^2c - 3bc^2 - 6bcxy$.
Ans. $b - 3c - 6xy$.
6. $4a^2c - 4acx$ and $3a^2g - 3agx$. *Ans.* $a(a - x)$, or $a^2 - ax$.
7. $4c^2 - 12cx + 9x^2$ and $4c^2 - 9x^2$. *Ans.* $2c - 3x$.
8. $x^3 - y^3$ and $x^2 - y^2$. *Ans.* $x - y$.
9. $4c^2 + 4bc + b^2$ and $4c^2 - b^2$. *Ans.* $2c + b$.
10. $25a^2c^2 - 9x^4y^4$ and $5acd^2 + 3d^2x^2y^2$. *Ans.* $5ac + 3x^2y^2$.

71 a. When the quantities cannot be readily factored, another method of finding the G. C. D. is used, — one of successive division, depending on the following principles: —

(1) *Any* COMMON *factor of two quantities is a factor of their G. C. D.: hence, in finding the G. C. D., any common factor that is apparent may be suppressed, and set aside as a factor of the G. C. D.*

(2) *No factor that is* NOT *common can be a factor of the G. C. D.: hence any factor in either polynomial that is not common may be suppressed and disregarded. So, also, either polynomial may be multiplied by any factor that is not contained in the other.*

(3) *Any quantity that will exactly divide two other quantities will divide the difference between any two multiples of those quantities.*

Thus, $a-b$ will exactly divide the two quantities

$$a^2-2ab+b^2 \text{ and } a^2-b^2.$$

Multiply each of these quantities by some number (that is, take *multiples* of them), — the first by 5, and the second by 3, giving $5a^2-10ab+5b^2$ and $3a^2-3b^2$. The difference between these multiples is $2a^2-10ab+8b^2$, and this is exactly divisible by $a-b$.

(4) *Any quantity that will exactly divide the difference between any multiples of two quantities and one of the quantities will exactly divide the other quantity.*

Thus, take the two quantities

$$c^3+c^2d+cd^2+d^3 \text{ and } c^2+2cd+d^2.$$

Take a multiple of each, multiplying the first by 6, and the second by $2c$, giving

$$6c^3+6c^2d+6cd^2+6d^3 \text{ and } 2c^3+4c^2d+2cd^2.$$

The difference of these multiples, $4c^3+2c^2d+4cd^2+6d^3$, is divisible by $c+d$, and so is the quantity $c^2+2cd+d^2$; $c+d$ will also divide the other quantity, $c^3+c^2d+cd^2+d^3$.

To show how these principles may be applied, take the two polynomials

$$18x^3-18x^2y+6xy^2-6y^3 \text{ and } 12x^2-15xy+3y^2.$$

These are arranged according to the descending powers of the same letter, x.

The factor 3 may be suppressed in each, and, as it is common, is to be set aside as a factor of the G. C. D. (Principle 1), leaving

$$6x^3 - 6x^2y + 2xy^2 - 2y^3 \text{ and } 4x^2 - 5xy + y^2.$$

Multiply the first of these by 2 (Principle 2), to make its first term exactly divisible by the first term of the second polynomial, and avoid fractional coefficients, and divide the result by the second polynomial. In performing this division, multiply the first remainder by 4 to avoid fractional coefficients.

$$\begin{array}{r|l}
12x^3 - 12x^2y + 4xy^2 - 4y^3 & 4x^2 - 5xy + y^2 \\
\underline{12x^3 - 15x^2y + 3xy^2 } & 3x + 3y \\
3x^2y + xy^2 - 4y^3 & \\
12x^2y + 4xy^2 - 16y^3 & \\
\underline{12x^2y - 15xy^2 + 3y^3} & \\
19xy^2 - 19y^3 &
\end{array}$$

$19xy^2 - 19y^3$ is a difference between multiples of

$$12x^3 - 12x^2y + 4xy^2 - 4y^3 \text{ and } 4x^2 - 5xy + y^2.$$

Any quantity that will divide these latter quantities must divide $19xy^2 - 19y^3$ (Principle 3), and any quantity that will divide

$$19xy^2 - 19y^3 \text{ and } 4x^2 - 5xy + y^2$$

must also divide $12x^3 - 12x^2y + 4xy^2 - 4y^3$. What, then, is the G. C. D. of $19xy^2 - 19y^3$ and $4x^2 - 5xy + y^2$?

$19y$ may be suppressed and disregarded (Principle 2).

$$\begin{array}{r|l}
4x^2 - 5xy + y^2 & x - y \\
\underline{4x^2 - 4xy } & 4x - y \\
- xy + y^2 & \\
\underline{- xy + y^2} &
\end{array}$$

$x - y$ is the G. C. D. of $19xy^2 - 19y^3$ and $4x^2 - 5xy + y^2$; and hence it is the G. C. D. of $12x^3 - 12x^2y + 4xy^2 - 4y^3$ and $4x^2 - 5xy + y^2$. This multiplied by 3, the common factor set aside in the beginning, giving $3x - 3y$, will be the G. C. D. of the polynomials given.

As all other examples of the same kind can be performed in a like way, we have the following rule: —

Arrange the polynomials with reference to the same letter, and suppress all monomial factors in either polynomial. If any factor so suppressed is common to the two polynomials, set it aside as a factor of the greatest common divisor.

Multiply the polynomial containing the highest power of the leading letter thus prepared by such a factor as will make its first term exactly divisible by the first term of the other. Divide the first by the second, and continue the division until the greatest exponent of the leading letter, in the remainder, is at least one less than in the divisor. If the highest power of the leading letter in both polynomials is the same, either polynomial may be used as the divisor.

Take the divisor as a new dividend and this remainder as a divisor, and proceed as before; and continue until a remainder 0, *or one independent of the leading letter, is found. In the first case, the last divisor, multiplied by common factors set aside, will be the greatest common divisor; in the second, the product of factors set aside, if any, will be the greatest common divisor. If there be no remainder* 0, *and no factors to set aside, the polynomials are prime with respect to each other, and have, of course, no common divisor.*

(1) Find the G. C. D. of

$$3x^3-13x^2+23x-21 \text{ and } 12x^3+2x^2-88x+42.$$

Suppressing the factor 2 in the second polynomial, and introducing the factor 2 into the first, we have

$$\begin{array}{r|l} 6x^3-26x^2+46x-42 & 6x^3+x^2-44x+21 \\ \cline{2-2} 6x^3+x^2-44x+21 & 1 \\ \cline{1-1} -27x^2+90x-63 & \end{array}$$

Suppressing the factor -9 in the first remainder, and proceeding as before, we have

$$\begin{array}{l|l}
6x^3 + \quad x^2 - 44x + 21 & 3x^2 - 10x + 7 \\
\cline{2-2}
6x^3 - 20x^2 + 14x & 2x + 7 \\
\cline{1-1}
\quad\quad 21x^2 - 58x + 21 & \\
\quad\quad 21x^2 - 70x + 49 & \\
\cline{1-1}
\quad\quad\quad\quad 12x - 28 &
\end{array}$$

Suppressing the factor 4 in the second remainder, and proceeding as before, we have

$$\begin{array}{l|l}
3x^2 - 10x + 7 & 3x - 7 \\
\cline{2-2}
3x^2 - \quad 7x & x - 1 \\
\cline{1-1}
\quad\quad - \quad 3x + 7 & \\
\quad\quad - \quad 3x + 7 & \\
\cline{1-1}
\quad\quad\quad\quad 0 &
\end{array}$$

Hence $3x - 7$ is the G. C. D. sought.

(2) $x^4 - 7x^3 + 8x^2 + 28x - 48$ and $x^3 - 8x^2 + 19x - 14$.
Ans. $x - 2$.

(3) $4x^4 + 9x^3 + 2x^2 - 2x - 4$ and $3x^3 + 5x^2 - x + 2$.
Ans. $x + 2$.

(4) $x^5 + 3x^4 - 8x^2 - 9x - 3$ and $x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6$.
Ans. $(x + 1)(x + 1)(x + 1)$.

(5) $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$ and $4x^4 + 2x^3 - 18x^2 + 3x - 5$.
Ans. $2x^3 - 4x^2 + x - 1$.

(6) $20x^6 - 12x^5 + 16x^4 - 15x^3 + 14x^2 - 15x + 4$ and $15x^4 - 9x^3 + 47x^2 - 21x + 28$.
Ans. $5x^2 - 3x + 4$.

(7) $x^5 - x^4 - 3x^3 - 2x^2 + 14x + 21$, $x^4 + x^2 - 2x + 6$.
Ans. $x^2 + 2x + 3$.

To find the G. C. D. of three or more quantities,

Find the G. C. D. of the first and second, then the G. C. D. of this result and the third quantity, and so on to the last.

Find the G. C. D. of the following: —

(8) $4ax^2y$, $16abx^2$, and $24acx^2$.

(9) $3x^2-6x$, $2x^3-4x^2$, and x^2y-2xy.

(10) $x^3-9x^2+26x-24$, $x^3-10x^2+31x-30$, and $x^3-11x^2+38x-40$. *Ans.* $x-2$.

(11) x^4-10x^2+9, $x^4+10x^3+20x^2-10x-21$, and $x^4+4x^3-22x^2-4x+21$. *Ans.* x^2-1.

(12) x^4-a^4, x^3+a^3, and x^2-a^2. *Ans.* $x+a$.

(13) a^5+b^5, a^4-b^4, and a^3+b^3. *Ans.* $a+b$.

(14) $3x^3-7x^2y+5xy^2-y^3$, $x^2y+3xy^2-3x^3-y^3$, and $3x^3+5x^2y+xy^2-y^3$. *Ans.* $3x-y$.

Least Common Multiple.

72. A **multiple** of a number is any quantity that can be divided by that number without a remainder. Thus, $8a^2b$ is a multiple of 8, also of a^2 and of b.

73. A **common multiple** of two or more quantities is a quantity that can be divided by each separately without a remainder. Thus, $24a^3x^3$ is a common multiple of $6ax$ and $4a^2x$.

74. The **least common multiple** of two or more quantities is the simplest quantity that can be divided by each without a remainder. Thus, $12a^2b^2x^2$ is the least common multiple of $2a^2x$, $4ab^2$, and $6a^2b^2x^2$.

75. Since the common multiple is a dividend of each of the quantities, and since the division is exact, the common multiple must contain every prime factor in all the quantities; and, if the same factor enters more than once, it must enter an equal number of times into the common multiple.

When the given quantities can be factored by any of the methods already given, the least common multiple may be found by the following rule: —

Resolve each of the quantities into its prime factors.

Take each factor as many times as it enters any one of the quantities, and form the continued product of these factors. It will be the least common multiple.

Exercises.

NOTE. — For convenience, L. C. M. will be used to denote least common multiple.

1. Find the L. C. M. of $12a^3b^2c^2$ and $8a^2b^3$.

$$12a^3b^2c^2 = 2.2.3.aaabbcc.$$
$$8a^2b^3 = 2.2.2.aabbb.$$

Now, since 2 enters three times as a factor, it must enter three times in the common multiple; 3 must enter once; a, three times; b, three times; and c, twice: hence $2.2.2.3\,aaabbbcc$, or $24a^3b^3c^2$, is the L. C. M.

Find the L. C. M. of

2. $6a$, $5a^2b$, and $25abc^2$. *Ans.* $150a^2bc^2$.

3. $3a^2b$, $9abc$, and $27a^2x^2$. *Ans.* $27a^2bcx^2$.

4. $4a^2x^2y^2$, $8a^3xy$, $16a^4y^3$, and $24a^5y^4x$. *Ans.* $48a^5x^2y^4$.

5. $ax - bx$, $ay - by$, and x^2y^2.
Ans. $(a-b)x.x.yy = ax^2y^2 - bx^2y^2$.

6. $a+b$, a^2-b^2, and $a^2+2ab+b^2$. *Ans.* $(a+b)^2(a-b)$.

7. $3a^3b^2$, $9a^2x^2$, $18a^4y^3$, $3a^2y^2$. *Ans.* $18a^4b^2x^2y^3$.

8. $8a^2(a-b)$, $15a^5(a-b)^2$, and $12a^3(a^2-b^2)$.
Ans. $120a^5(a-b)^2(a+b)$.

75a. When the given quantities cannot readily be factored, another method is used.

The L. C. M. of two polynomials contains *all* the factors of each polynomial; the G. C. D. contains all the factors *common* to the two polynomials. Hence, to find the L. C. M. of two polynomials, we have the following rule: —

Find the G. C. D. of the two quantities. Divide one of the quantities by it, then multiply the other quantity by the quotient.

Exercises.

Find the L. C. M. of the following: —

1. $2x^2 - xy - 6y^2$ and $3x^2 - 8xy + 4y^2$.

Their G. C. D. is $x - 2y$: hence their L. C. M. is

$$\frac{2x^2 - xy - 6y^2}{x - 2y} \times (3x^2 - 8xy + 4y^2) = (2x + 3y)(3x^2 - 8xy + 4y^2).$$

2. $3x^2 - 5x + 2$ and $4x^3 - 4x^2 - x + 1$.
Ans. $(3x - 2)(4x^3 - 4x^2 - x + 1)$.

3. $6x^2 - x - 1$ and $2x^2 + 3x - 2$.
Ans. $(3x + 1)(2x^2 + 3x - 2)$.

To find the L. C. M. of several quantities,

Find the L. C. M. of the first and second, then of that result and the third, and so on to the last.

Find the L. C. M. of the following: —

4. $x^3 - 6x^2 + 11x - 6$, $x^3 - 9x^2 + 26x - 24$, and $x^3 - 8x^2 + 19x - 12$.
Ans. $(x - 1)(x - 2)(x - 3)(x - 4)$.

5. $x^4 - 10x^2 + 9$, $x^4 + 10x^3 + 20x^2 - 10x - 21$, and $x^4 + 4x^3 - 22x^2 - 4x + 21$.
Ans. $(x^2 - 1)(x^2 - 9)(x + 7)$.

6. $x^2-(a+b)x+ab$, $x^2-(b+c)x+bc$, and $x^2-(c+a)x+ca$. *Ans.* $(x-a)(x-b)(x-c)$.

7. $6(a^3-b^3)(a-b)^3$, $9(a^4-b^4)(a-b)^2$, and $12(a^2-b^2)^3$. *Ans.* $36(a^4-b^4)(a^3-b^3)(a^2-b^2)^2$.

8. x^2+5x+6, x^2-2x-8, and x^2-x-12. *Ans.* $(x+3)(x-4)(x+2)$.

9. $6x^2+13x+6$, $6x^2-5x-6$, and $4x^2-9$. *Ans.* $(3x+2)(2x+3)(2x-3)$.

10. $2x^2+11x+15$, $2x^2+x-10$, and x^2+x-6. *Ans.* $(2x+5)(x+3)(x-2)$.

CHAPTER IV.

FRACTIONS.

76. A **fractional unit** is any one of a number of equal parts of a unit. Thus, $\frac{1}{2}, \frac{1}{4}, \frac{1}{7}, \frac{1}{b}$, are fractional units.

77. A **fraction** is a fractional unit, or a collection of fractional units. Thus, $\frac{1}{2}, \frac{3}{4}, \frac{5}{7}, \frac{a}{b}$, are fractions.

78. Every fraction is composed of two parts, — the **denominator** and the **numerator.** The denominator shows into how many equal parts the unit 1 is divided; and the numerator, how many of these parts are taken. Thus, in the fraction $\frac{a}{b}$, the denominator b shows that 1 is divided into b equal parts, and the numerator a shows that a of these parts are taken. The fractional unit, in all cases, is equal to the reciprocal of the denominator.

79. An **entire quantity** is one which contains no fractional part. Thus, 7, 11, a^3x, $4x^2 - 3y$, are entire quantities.

An entire quantity may be regarded as a fraction whose denominator is 1. Thus, $7 = \frac{7}{1}$, $ab = \frac{ab}{1}$.

80. A **mixed quantity** is a quantity containing both entire and fractional parts. Thus, $7\frac{4}{10}$, $8\frac{3}{7}$, $a + \frac{bx}{c}$, are mixed quantities.

81. Let $\frac{a}{b}$ denote any fraction, and q any quantity whatever. From the preceding definitions, $\frac{a}{b}$ denotes that $\frac{1}{b}$ is taken a times; also $\frac{aq}{b}$ denotes that $\frac{1}{b}$ is taken aq times, that is,

$$\frac{aq}{b}=\frac{a}{b}\times q.$$

Hence

Multiplying the numerator of a fraction by any quantity is equivalent to multiplying the fraction by that quantity.

We see, also, that

Any quantity may be multiplied by a fraction by multiplying it by the numerator, and then dividing the result by the denominator.

82. It is a principle of division that the same result will be obtained if we divide the quantity a by the product of two factors, $p \times q$, as would be obtained by dividing it first by one of the factors, p, and then dividing that result by the other factor, q; that is,

$$\frac{a}{pq}=\left(\frac{a}{p}\right)\div q; \text{ or } \frac{a}{pq}=\left(\frac{a}{q}\right)\div p.$$

Hence

Multiplying the denominator of a fraction by any quantity is equivalent to dividing the fraction by that quantity.

83. Since the operations of multiplication and division are the converse of each other, it follows, from the preceding principles, that

Dividing the numerator of a fraction by any quantity is equivalent to dividing the fraction by that quantity.

Dividing the denominator of a fraction by any quantity is equivalent to multiplying the fraction by that quantity.

84. Since a quantity may be multiplied and the result divided by the same quantity without altering the value, it follows that

Both terms of a fraction may be multiplied by any quantity, or both divided by any quantity, without changing the value of the fraction.

TRANSFORMATION OF FRACTIONS.

85. The **transformation** of a quantity is the operation of changing its form without altering its value. The term "reduce" has a technical signification, and means "to transform."

86. To reduce an Entire Quantity to a Fractional Form having a Given Denominator.

Let a be the quantity, and b the given denominator. We have, evidently, $a = \frac{ab}{b}$. Hence the rule: —

Multiply the quantity by the given denominator, and write the product over this given denominator.

87. To reduce a Fraction to its Lowest Terms.

A fraction is in its *lowest terms* when the numerator and the denominator contain no common factors.

It has been shown that both terms of a fraction may be divided by the same quantity without altering its value: therefore, if they have any common factors, we may strike them out. Hence the rule: —

Resolve each term of the fraction into its prime factors, then strike out all that are common to both.

The same result is attained by dividing both terms of the fraction by any quantity that will divide them without a remainder, or by dividing them by their G. C. D.

Exercises.

Reduce to their lowest terms

1. $\dfrac{15a^2c^2}{25acd}$.

Factoring, $$\frac{15a^2c^2}{25acd}=\frac{3.5aacc}{5.5acd}.$$

Canceling the common factors, 5, a, and c, we have

$$\frac{15a^2c^2}{25acd}=\frac{3ac}{5d}.$$

2. $\dfrac{85b^7cd^5}{15b^7c^8d^5}$. *Ans.* $\dfrac{17}{3c^7}$.

3. $\dfrac{60c^6d^4f^5}{12c^5d^8f^9}$. *Ans.* $\dfrac{5c}{d^4f^4}$.

4. $\dfrac{ab-ac}{b-c}$. *Ans.* $\dfrac{a}{1}=a$.

5. $\dfrac{n^2-2n+1}{n^2-1}$. *Ans.* $\dfrac{n-1}{n+1}$.

6. $\dfrac{x^3-ax^2}{x^2-2ax+a^2}$. *Ans.* $\dfrac{x^2}{x-a}$.

7. $\dfrac{96a^3b^2c}{-12a^3b^2c}$. *Ans.* $-\dfrac{8}{1}=-8$.

8. $\dfrac{24b^5-36ab^4}{48a^4b^4-66a^5b^6}$. *Ans.* $\dfrac{4b-6a}{8a^4-11a^5b^2}$.

9. $\dfrac{a^2-b^2}{a^2-2ab+b^2}$. *Ans.* $\dfrac{a+b}{a-b}$.

10. $\dfrac{5a^3-10a^2b+5ab^2}{8a^3-8a^2b}$. *Ans.* $\dfrac{5(a-b)}{8a}$.

11. $\dfrac{3a^2+6a^2b^2}{12a^4+6a^3c^2}$. *Ans.* $\dfrac{1+2b^2}{4a^2+2ac^2}$.

12. $\dfrac{a^2+2ax+x^2}{3(a^2-x^2)}$. *Ans.* $\dfrac{a+x}{3(a-x)}$.

88. To reduce a Fraction to a Mixed Quantity.

When any term of the numerator is divisible by any term of the denominator, the transformation can be effected by division. Hence the rule: —

Perform the indicated division, continuing the operation as far as possible; then write the remainder over the denominator, and annex the result to the quotient found.

Exercises.

Reduce

1. $\dfrac{ax - a^2}{x}$. *Ans.* $a - \dfrac{a^2}{x}$.

2. $\dfrac{ax - x^2}{x}$. *Ans.* $a - x$.

3. $\dfrac{ab - 2a^2}{b}$. *Ans.* $a - \dfrac{2a^2}{b}$.

4. $\dfrac{a^2 - x^2}{a - x}$. *Ans.* $a + x$.

5. $\dfrac{x^3 - y^3}{x - y}$. *Ans.* $x^2 + xy + y^2$.

6. $\dfrac{10x^2 - 5x + 3}{5x}$. *Ans.* $2x - 1 + \dfrac{3}{5x}$.

7. $\dfrac{36x^3 - 72x + 32a^2x^2}{9x}$. *Ans.* $4x^2 - 8 + \dfrac{32a^2x}{9}$.

8. $\dfrac{18acf - 6bdcf - 2ad}{3adf}$. *Ans.* $\dfrac{6c}{d} - \dfrac{2bc}{a} - \dfrac{2}{3f}$.

9. $\dfrac{x^2 + x - 4}{x + 2}$. *Ans.* $x - 1 - \dfrac{2}{x + 2}$.

10. $\dfrac{a^2 + b^2}{a + b}$. *Ans.* $a - b + \dfrac{2b^2}{a + b}$.

11. $\dfrac{x^2 + 3x - 25}{x - 4}$. *Ans.* $x + 7 + \dfrac{3}{x - 4}$.

89. To reduce a Mixed Quantity to a Fractional Form.

This transformation is the converse of the preceding, and may be effected by the following rule: —

Multiply the entire part by the denominator of the fraction, and add to the product the numerator. Write the result over the denominator of the fraction.

Exercises.

Reduce the following to fractional forms: —

1. $6\frac{1}{7}$.

$$6 \times 7 = 42;\ 42 + 1 = 43: \text{ hence } 6\tfrac{1}{7} = \frac{43}{7}.$$

2. $x - \dfrac{a^2 - x^2}{x} = \dfrac{x^2 - (a^2 - x^2)}{x}.$ *Ans.* $\dfrac{2x^2 - a^2}{x}.$

3. $x - \dfrac{ax + x^2}{2a}.$ *Ans.* $\dfrac{ax - x^2}{2a}.$

4. $5 + \dfrac{2x - 7}{3x}.$ *Ans.* $\dfrac{17x - 7}{3x}.$

5. $1 - \dfrac{x - a - 1}{a}.$ *Ans.* $\dfrac{2a - x + 1}{a}.$

6. $1 + 2x - \dfrac{x - 3}{5x}.$ *Ans.* $\dfrac{10x^2 + 4x + 3}{5x}.$

7. $2a + b - \dfrac{3c + 4}{8}.$ *Ans.* $\dfrac{16a + 8b - 3c - 4}{8}.$

8. $6ax + b - \dfrac{6a^2x - ab}{4a}.$ *Ans.* $\dfrac{18a^2x + 5ab}{4a}.$

9. $8 + 3ab - \dfrac{8 + 6a^2b^2x^4}{12abx^4}.$ *Ans.* $\dfrac{96abx^4 + 30a^2b^2x^4 - 8}{12abx^4}.$

90. To reduce Fractions having Different Denominators to Equivalent Fractions having the Least Common Denominator.

This transformation is effected by finding the L. C. M. of the denominators.

Reduce $\frac{1}{3}$, $\frac{3}{4}$, and $\frac{5}{12}$ to their least common denominator.

The L. C. M. of the denominators is 12, which is also the least common denominator of the required fractions. If each fraction be multiplied by 12, and the result divided by 12, the values of the fractions will not be changed.

$\frac{1}{3} \times 12 = 4$, 1st new numerator.

$\frac{3}{4} \times 12 = 9$, 2d new numerator.

$\frac{5}{12} \times 12 = 5$, 3d new numerator.

Hence $\frac{4}{12}$, $\frac{9}{12}$, and $\frac{5}{12}$ are the new equivalent fractions.

From the preceding example we deduce the following rule: —

Find the least common multiple of the denominators.

Multiply each fraction by it, and cancel the denominator.

Write each product over the common multiple, and the results will be the required fractions.

Or, in general,

Multiply each numerator by all the denominators except its own, for the new numerators; and all the denominators together, for a common denominator.

Exercises.

Reduce the following to their least common denominators:—

1. $\frac{a}{a^2-b^2}$ and $\frac{c}{a+b}$.

The L. C. M. of the denominators is $(a+b)(a-b)$.

$$\frac{a}{a^2-b^2}\times(a+b)(a-b)=a.$$

$$\frac{c}{a+b}\times(a+b)(a-b)=c(a-b).$$

Hence $\frac{a}{(a+b)(a-b)}$ and $\frac{c(a-b)}{(a+b)(a-b)}$ are the required fractions.

2. $\frac{3x}{4}$, $\frac{4}{6}$, and $\frac{12x^2}{15}$. *Ans.* $\frac{45x}{60}$, $\frac{40}{60}$, $\frac{48x^2}{60}$.

3. a, $\frac{3b^2}{4}$, and $\frac{5c^3}{6}$. *Ans.* $\frac{12a}{12}$, $\frac{9b^2}{12}$, $\frac{10c^3}{12}$.

4. $\frac{3x}{2a}$, $\frac{2b}{3c}$, and d. *Ans.* $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, $\frac{6acd}{6ac}$.

5. $\frac{3}{4}$, $\frac{2x}{3}$, $a+\frac{2x}{a}$. *Ans.* $\frac{9a}{12a}$, $\frac{8ax}{12a}$, $\frac{12a^2+24x}{12a}$.

6. $\frac{x}{1-x}$, $\frac{x^2}{(1-x)^2}$, and $\frac{x^3}{(1-x)^3}$.

Ans. $\frac{x(1-x)^2}{(1-x)^3}$, $\frac{x^2(1-x)}{(1-x)^3}$, and $\frac{x^3}{(1-x)^3}$.

7. $\frac{c}{5a}$, $\frac{c-b}{c}$, and $\frac{c}{a+b}$.

Ans. $\frac{ac^2+bc^2}{5a^2c+5abc}$, $\frac{5a^2c-5a^2b+5abc-5ab^2}{5a^2c+5abc}$, $\frac{5ac^2}{5a^2c+5abc}$.

8. $\frac{cx}{a-x}$, $\frac{dx^2}{a+x}$, and $\frac{x^3}{a+x}$.

Ans. $\frac{cx(a+x)}{a^2-x^2}$, $\frac{dx^2(a-x)}{a^2-x^2}$, and $\frac{x^3(a-x)}{a^2-x^2}$.

Addition of Fractions.

91. Fractions can be added only when they have a common unit; that is, when they have a common denominator. In that case, the sum of the numerators will indicate how many times that unit is taken in the entire collection. Hence the rule: —

Reduce the fractions to be added, to a common denominator.
Add the numerators together, for a new numerator, and write the sum over the common denominator.

Exercises.

1. Add $\frac{6}{2}$, $\frac{4}{3}$, and $\frac{2}{5}$.

By reducing to a common denominator, we have

$6 \times 3 \times 5 = 90$, 1st numerator.

$4 \times 2 \times 5 = 40$, 2d numerator.

$2 \times 3 \times 2 = 12$, 3d numerator.

$2 \times 3 \times 5 = 30$, the denominator.

Hence the expression for the sum of the fractions becomes

$$\frac{90}{30} + \frac{40}{30} + \frac{12}{30} = \frac{142}{30},$$

which, being reduced to the simplest form, gives $4\frac{11}{15}$.

2. Find the sum of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$.

Here $\left.\begin{array}{l} a \times d \times f = adf \\ c \times b \times f = cbf \\ e \times b \times d = ebd \end{array}\right\}$, the new numerators,

and $b \times d \times f = bdf$, the common denominator.

Hence $\frac{adf}{bdf} + \frac{cbf}{bdf} + \frac{ebd}{bdf} = \frac{adf + cbf + ebd}{bdf}$, the sum.

Add the following: —

3. $a-\frac{3x^2}{b}$ and $b+\frac{2ax}{c}$. *Ans.* $a+b+\frac{2abx-3cx^2}{bc}$.

4. $\frac{x}{2}$, $\frac{x}{3}$, and $\frac{x}{4}$. *Ans.* $x+\frac{x}{12}$.

5. $\frac{x-2}{3}$ and $\frac{4x}{7}$. *Ans.* $\frac{19x-14}{21}$.

6. $x+\frac{x-2}{3}$ and $3x+\frac{2x-3}{4}$. *Ans.* $4x+\frac{10x-17}{12}$.

7. $4x$, $\frac{5x^2}{2a}$, and $\frac{x+a}{2x}$. *Ans.* $4x+\frac{5x^3+ax+a^2}{2ax}$.

8. $\frac{2x}{3}$, $\frac{7x}{4}$, and $\frac{2x+1}{5}$. *Ans.* $2x+\frac{49x+12}{60}$.

9. $4x$, $\frac{7x}{9}$, and $2+\frac{x}{5}$. *Ans.* $2+4x+\frac{44x}{45}$.

10. $3x+\frac{2x}{5}$ and $x-\frac{8x}{9}$. *Ans.* $3x+\frac{23x}{45}$.

11. $ac-\frac{6b}{8a}$ and $1-\frac{c}{d}$. *Ans.* $1+ac-\frac{6bd+8ac}{8ad}$.

12. $\frac{3}{(x-1)^3}$, $\frac{3}{(x-1)^2}$, and $\frac{4}{x-1}$. *Ans.* $\frac{4x^2-5x+4}{(x-1)^3}$.

13. $\frac{1}{4(1+a)}$, $\frac{1}{4(1-a)}$, and $\frac{1}{2(1-a^2)}$. *Ans.* $\frac{1}{1-a^2}$.

Subtraction of Fractions.

92. Fractions can be subtracted only when they have the same unit; that is, a common denominator. In that case, the numerator of the minuend, *minus* that of the subtrahend, will indicate the number of times that the common unit is to be taken in the difference. Hence the rule: —

Reduce the two fractions to a common denominator.

Then subtract the numerator of the subtrahend from that of the minuend, for a new numerator, and write the remainder over the common denominator.

Exercises.

1. What is the difference between $\frac{3}{7}$ and $\frac{2}{8}$?

$$\frac{3}{7}-\frac{2}{8}=\frac{24}{56}-\frac{14}{56}=\frac{10}{56}=\frac{5}{28}.$$

2. Find the difference of the fractions $\frac{x-a}{2b}$ and $\frac{2a-4x}{3c}$.

Here $\left.\begin{array}{l}(x-a)\times 3c=3cx-3ac \\ (2a-4x)\times 2b=4ab-8bx\end{array}\right\}$, the numerators;

and $2b\times 3c=6bc$, the common denominator.

Hence $\frac{3cx-3ac}{6bc}-\frac{4ab-8bx}{6bc}=\frac{3cx-3ac-4ab+8bx}{6bc}.$

3. Required the difference of $\frac{12x}{7}$ and $\frac{3x}{5}$. *Ans.* $\frac{39x}{35}$.

4. Required the difference of $5y$ and $\frac{3y}{8}$. *Ans.* $\frac{37y}{8}$.

5. Required the difference of $\frac{3x}{7}$ and $\frac{2x}{9}$. *Ans.* $\frac{13x}{63}$.

6. From $\frac{x+y}{x-y}$ subtract $\frac{x-y}{x+y}$. *Ans.* $\frac{4xy}{x^2-y^2}$.

7. From $\frac{1}{y-z}$ subtract $\frac{1}{y^2-z^2}$. *Ans.* $\frac{y+z-1}{y^2-z^2}$.

Find the differences of the following: —

8. $\frac{8x+a}{5b}$ and $\frac{2x+7}{8}$. *Ans.* $\frac{24x+8a-10bx-35b}{40b}$.

9. $3x+\frac{x}{b}$ and $x-\frac{x-a}{c}$. *Ans.* $2x+\frac{cx+bx-ab}{bc}$.

10. $a+\frac{a-x}{a(a+x)}$ and $\frac{a+x}{a(a-x)}$. *Ans.* $a-\frac{4x}{a^2-x^2}$.

Multiplication of Fractions.

93. Let $\frac{a}{b}$ and $\frac{c}{d}$ represent any two fractions. It has been shown (§ 81) that any quantity may be multiplied by a fraction by first multiplying by the numerator, and then dividing the result by the denominator.

To multiply $\frac{a}{b}$ by $\frac{c}{d}$, we first multiply by c, giving $\frac{ac}{b}$; then we divide this result by d, which is done by multiplying the denominator by d. This gives for the product, $\frac{ac}{bd}$; that is,

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$

Hence the rule: —

If there are mixed quantities, reduce them to a fractional form; then

Multiply the numerators together, for a new numerator; and the denominators, for a new denominator.

Exercises.

1. Multiply $a+\frac{bx}{a}$ by $\frac{c}{d}$.

First, $$a+\frac{bx}{a}=\frac{a^2+bx}{a};$$

hence $$\frac{a^2+bx}{a}\times\frac{c}{d}=\frac{a^2c+bcx}{ad}.$$

Find the products of the following quantities: —

2. $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{3ac}{2b}$. *Ans.* $9ax$.

3. $b+\frac{bx}{a}$ and $\frac{a}{x}$. *Ans.* $\frac{ab+bx}{x}$.

4. $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$. *Ans.* $\frac{x^4-b^4}{b^2c+bc^2}$.

5. $x+\dfrac{x+1}{a}$ and $\dfrac{x-1}{a+b}$. *Ans.* $\dfrac{ax^2-ax+x^2-1}{a^2+ab}$.

6. $a+\dfrac{ax}{a-x}$ and $\dfrac{a^2-x^2}{x+x^2}$. *Ans.* $\dfrac{a^3+a^2x}{x+x^2}$.

Multiply

7. $\dfrac{2a}{a-b}$ by $\dfrac{a^2-b^2}{3}$.

We have, by the rule,

$$\frac{2a}{a-b}\times\frac{a^2-b^2}{3}=\frac{2a(a^2-b^2)}{3(a-b)}=\frac{2a(a+b)(a-b)}{3(a-b)}=\frac{2a}{3}(a+b).$$

NOTE. — After indicating the operation, we factored both numerator and denominator, and then canceled the common factors, before performing the multiplication. *This should be done whenever there are common factors.*

8. $\dfrac{2}{x-y}$ by $\dfrac{x^2-y^2}{a}$. *Ans.* $\dfrac{2(x+y)}{a}$.

9. $\dfrac{x^2-4}{3}$ by $\dfrac{4x}{x+2}$. *Ans.* $\dfrac{4x(x-2)}{3}$.

10. $\dfrac{(a+b)^2}{2x}$ by $\dfrac{4x^2}{(a+b)}$. *Ans.* $2x(a+b)$.

11. $\dfrac{(x-1)^2}{y^3}$ by $\dfrac{(x+1)y^2}{x-1}$. *Ans.* $\dfrac{x^2-1}{y}$.

12. $\dfrac{(a^2-x^2)}{1-x^2}$ by $\dfrac{1+x}{a+x}$. *Ans.* $\dfrac{a-x}{1-x}$.

13. $x+\dfrac{2xy}{x-y}$ by $x-\dfrac{2xy}{x+y}$. *Ans.* x^2.

14. $\dfrac{2a-b}{4a}$ by $\dfrac{6a-2b}{b^2-2ab}$. *Ans.* $\dfrac{b-3a}{2ab}$.

15. $x-\dfrac{y^2}{x}$ by $\dfrac{x}{y}+\dfrac{y}{x}$. *Ans.* $\dfrac{x^4-y^4}{x^2y}$.

Division of Fractions.

94. Since $\frac{p}{q} = p \times \frac{1}{q}$, it follows that dividing by a quantity is equivalent to multiplying by its reciprocal. But the reciprocal of a fraction, $\frac{c}{d}$, is $\frac{d}{c}$ (§ 28): consequently, to divide any quantity by a fraction, we invert the terms of the divisor, and multiply by the resulting fraction. Hence

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

Whence the following rule for dividing one fraction by another: —

Reduce mixed quantities to fractional forms.

Invert the terms of the divisor, and multiply the dividend by the resulting fraction.

Note. — The same remarks as were made on *factoring* and *reducing*, under the head of "Multiplication," are applicable in division.

Exercises.

Divide

1. $a - \frac{b}{2c}$ by $\frac{f}{g}$.

$$a - \frac{b}{2c} = \frac{2ac - b}{2c}.$$

Hence $$a - \frac{b}{2c} \div \frac{f}{g} = \frac{2ac - b}{2c} \times \frac{g}{f} = \frac{2acg - bg}{2cf}.$$

2. $\frac{2(x+y)}{a}$ by $\frac{x^2 - y^2}{a}$.

$$\frac{2(x+y)}{a} \times \frac{a}{x^2 - y^2} = \frac{2(x+y)}{a} \times \frac{a}{(x+y)(x-y)} = \frac{2}{x-y}.$$

3. $\frac{7x}{5}$ by $\frac{12}{13}$. *Ans.* $\frac{91x}{60}$.

4. $\frac{4x^2}{7}$ by $5x$. *Ans.* $\frac{4x}{35}$.

5. $\dfrac{x+1}{6}$ by $\dfrac{2x}{3}$. *Ans.* $\dfrac{x+1}{4x}$.

6. $\dfrac{x}{x-1}$ by $\dfrac{x}{2}$. *Ans.* $\dfrac{2}{x-1}$.

7. $\dfrac{5x}{3}$ by $\dfrac{2a}{3b}$. *Ans.* $\dfrac{5bx}{2a}$.

8. $\dfrac{x-b}{8cd}$ by $\dfrac{3cx}{4d}$. *Ans.* $\dfrac{x-b}{6c^2x}$.

9. $\dfrac{4x^2-8x}{3}$ by $\dfrac{x^2-4}{3}$. *Ans.* $\dfrac{4x}{x+2}$.

10. $\dfrac{x^4-b^4}{x^2-2bx+b^2}$ by $\dfrac{x^2+bx}{x-b}$. *Ans.* $x+\dfrac{b^2}{x}$.

11. $2x(a+b)$ by $\dfrac{4x^2}{a+b}$. *Ans.* $\dfrac{(a+b)^2}{2x}$.

12. $\dfrac{x^2-1}{y}$ by $\dfrac{(x+1)y^2}{x-1}$. *Ans.* $\dfrac{(x-1)^2}{y^3}$.

13. $\dfrac{a^2-ax}{bc+bx}$ by $\dfrac{3(c-x)}{4(a+x)}$. *Ans.* $\dfrac{4a(a^2-x^2)}{3b(c^2-x^2)}$.

14. $\dfrac{a-x}{1-x}$ by $\dfrac{1+x}{a+x}$. *Ans.* $\dfrac{a^2-x^2}{1-x^2}$.

15. x^2 by $x-\dfrac{2xy}{x+y}$. *Ans.* $\dfrac{x^2+xy}{x-y}$.

16. $\dfrac{b-3a}{2ab}$ by $\dfrac{6a-2b}{b^2-2ab}$. *Ans.* $\dfrac{2a-b}{4a}$.

17. $\dfrac{x^4-y^4}{x^2y}$ by $\dfrac{x}{y}+\dfrac{y}{x}$. *Ans.* $\dfrac{x^2-y^2}{x}$.

18. $m^2+1+\dfrac{1}{m^2}$ by $m+\dfrac{1}{m}+1$. *Ans.* $m+\dfrac{1}{m}-1$.

19. $\left(x+\dfrac{y-x}{1+xy}\right)$ by $\left(1-x\dfrac{y-x}{1+xy}\right)$. *Ans.* y.

20. $\left(\dfrac{x+2y}{x+y}+\dfrac{x}{y}\right)$ by $\left(\dfrac{x+2y}{y}-\dfrac{x}{x+y}\right)$. *Ans.* 1.

CHAPTER V.

EQUATIONS OF THE FIRST DEGREE, AND INEQUALITIES.

EQUATIONS OF THE FIRST DEGREE.

95. An **equation** is an expression of equality between two quantities. Thus, $x = b + c$ is an equation expressing the fact that the quantity x is equal to the sum of the quantities b and c.

96. Every equation is composed of two parts, connected by the sign of equality. These parts are called **members.** The part on the left of the sign of equality is called the **first member**; that on the right, the **second member.** Thus, in the equation $x + a = b - c$, $x + a$ is the first member; and $b - c$, the second member.

97. An equation of the **first degree** is one which involves only the first power of the unknown quantity. Thus,

$$6x + 3x - 5 = 13 \qquad (1)$$

$$ax + bx + c = d \qquad (2)$$

are equations of the first degree.

98. A **numerical equation** is one in which the coefficients of the unknown quantity are denoted by numbers.

99. A **literal equation** is one in which the coefficients of the unknown quantity are denoted by letters.

Equation (1) is a numerical equation; Equation (2) is a literal equation.

Transformation of Equations.

100. The **transformation** of an equation is the operation of changing its form without destroying the equality of its members.

101. An **axiom** is a self-evident proposition.

102. The transformation of equations depends upon the following axioms:—

Axiom 1. *If equal quantities be added to both members of an equation, the equality will not be destroyed.*

Axiom 2. *If equal quantities be subtracted from both members of an equation, the equality will not be destroyed.*

Axiom 3. *If both members of an equation be multiplied by the same quantity, the equality will not be destroyed.*

Axiom 4. *If both members of an equation be divided by the same quantity, the equality will not be destroyed.*

Axiom 5. *Like powers of the two members of an equation are equal.*

Axiom 6. *Like roots of the two members of an equation are equal.*

103. Two principal **transformations** are employed in the solution of equations of the first degree,—**clearing of fractions,** and **transposing.**

Take the equation

$$\frac{2x}{3}-\frac{3x}{4}+\frac{x}{6}=11.$$

The L. C. M. of the denominators is 12. If we multiply both members of the equation by 12, each term will reduce to an entire form, giving

$$8x-9x+2x=132.$$

Any equation may be reduced to entire terms in the same manner.

104. Hence, for clearing of fractions, we have the following rule: —

Find the L. C. M. of the denominators.

Multiply both members of the equation by it, reducing the fractional to entire terms.

NOTES. — 1. The reduction will be effected, if we divide the L. C. M. by each of the denominators, and then multiply the corresponding numerator by the quotient, dropping the denominator.

2. The transformation may be effected by multiplying each numerator into the product of all the denominators except its own, omitting denominators.

3. The transformation may also be effected by *multiplying both members of the equation by any multiple of the denominators.*

Exercises.

Clear the following equations of fractions: —

1. $\frac{x}{5}+\frac{x}{7}-4=3.$ *Ans.* $7x+5x-140=105.$

2. $\frac{x}{6}+\frac{x}{9}-\frac{x}{27}=8.$ *Ans.* $9x+6x-2x=432.$

3. $\frac{x}{2}+\frac{x}{3}-\frac{x}{9}+\frac{x}{12}=20.$ *Ans.* $18x+12x-4x+3x=720.$

4. $\frac{x}{5}+\frac{x}{7}-\frac{x}{2}=4.$ *Ans.* $14x+10x-35x=280.$

5. $\frac{x}{4}-\frac{x}{5}+\frac{x}{6}=15.$ *Ans.* $15x-12x+10x=900.$

6. $-\frac{x-4}{3}-\frac{x-2}{6}=\frac{5}{3}.$ *Ans.* $-2x+8-x+2=10.$

7. $\frac{x}{3-x}+4=\frac{3}{5}.$ *Ans.* $5x+60-20x=9-3x.$

8. $\frac{x}{4}-\frac{x}{6}+\frac{x}{8}+\frac{x}{9}=12.$ *Ans.* $18x-12x+9x+8x=864.$

9. $\frac{a}{b}-\frac{c}{d}+f=g.$ *Ans.* $ad-bc+bdf=bdg.$

10. $\frac{ax}{b}-\frac{2c^2x}{ab}+4a=\frac{4bc^2x}{a^3}-\frac{5a^3}{b^2}+\frac{2c^2}{a}-3b.$

The L. C. M. of the denominators is a^3b^2.

$$a^4bx-2a^2bc^2x+4a^4b^2=4b^3c^2x-5a^6+2a^2b^2c^2-3a^3b^3.$$

105. Transposition is the operation of changing a term from one member to the other without destroying the equality of the members.

Take, for example, the equation $5x-6=8+2x$.

If, in the first place, we subtract $2x$ from both members, the equality will not be destroyed, and we have

$$5x-6-2x=8.$$

Whence we see that the term $2x$, which was additive in the second member, becomes subtractive by passing into the first.

In the second place, if we add 6 to both members of the last equation, the equality will still exist, and we have

$$5x-6-2x+6=8+6;$$

or, since -6 and $+6$ cancel each other, we have

$$5x-2x=8+6.$$

Hence the term which was subtractive in the first member, passes into the second member with the sign of addition.

106. Therefore, for the transposition of the terms, we have the following rule: —

Any term may be transposed from one member of an equation to the other, if the sign be changed.

Exercises.

Transpose the unknown terms to the first member, and the known terms to the second, in the following: —

1. $3x + 6 - 5 = 2x - 7$. *Ans.* $3x - 2x = -7 - 6 + 5$.
2. $ax + b = d - cx$. *Ans.* $ax + cx = d - b$.
3. $4x - 3 = 2x + 5$. *Ans.* $4x - 2x = 5 + 3$.
4. $9x + c = cx - d$. *Ans.* $9x - cx = -d - c$.
5. $ax + f = dx + b$. *Ans.* $ax - dx = b - f$.
6. $6x - c = -ax + b$. *Ans.* $6x + ax = b + c$.

Solution of Equations.

107. The **solution** of an equation is the operation of finding such a value for the unknown quantity as will *satisfy* the equation; that is, such a value as, being substituted for the unknown quantity, will render the two members equal. This is called a **root** of the equation.

A **root** of an equation is said to be **verified**, when, being substituted for the unknown quantity in the given equation, the two members are found equal to each other.

Take the equation

$$\frac{3x}{2} - 4 = \frac{4(x-2)}{8} + 3.$$

Clearing of fractions (§ 104), and performing the operations indicated, we have

$$12x - 32 = 4x - 8 + 24.$$

Transposing all the unknown terms to the first member, and the known terms to the second (§ 106), we have

$$12x - 4x = -8 + 24 + 32.$$

Reducing the terms in the two members,

$$8x = 48.$$

Dividing both members by the coefficient of x,

$$x = \frac{48}{8} = 6.$$

VERIFICATION. $$\frac{3 \times 6}{2} - 4 = \frac{4(6-2)}{8} + 3,$$

or $$+9 - 4 = 2 + 3 = 5.$$

Hence 6 satisfies the equation, and therefore is a root.

108. By processes similar to the above, all equations of the first degree, containing but one unknown quantity, may be solved. Hence the rule: —

Clear the equation of fractions, and perform all the indicated operations.

Transpose all the unknown terms to the first member, and all the known terms to the second member.

Reduce all the terms in the first member to a single term, one factor of which shall be the unknown quantity, and the other factor will be the algebraic sum of its coefficients.

Divide both members by the coefficient of the unknown quantity: the second member will then be the value of the unknown quantity.

Exercises.

1. Solve the equation

$$\frac{5x}{12} - \frac{4x}{3} - 13 = \frac{7}{8} - \frac{13x}{6}.$$

Clearing of fractions,

$$10x - 32x - 312 = 21 - 52x.$$

Transposing, $$10x - 32x + 52x = 21 + 312.$$

Reducing, $$30x = 333.$$

Hence $$x = \frac{333}{30} = \frac{111}{10} = 11.1,$$

a result which may be verified by substituting it for x in the given equation.

2. Solve the equation

$$(3a - x)(a - b) + 2ax = 4b(x + a).$$

Performing the indicated operations, we have

$$3a^2 - ax - 3ab + bx + 2ax = 4bx + 4ab.$$

Transposing,

$$-ax + bx + 2ax - 4bx = 4ab + 3ab - 3a^2.$$

Reducing, $$ax - 3bx = 7ab - 3a^2.$$

Factoring, $$(a - 3b)x = 7ab - 3a^2.$$

Dividing both members by the coefficient of x,

$$x = \frac{7ab - 3a^2}{a - 3b}.$$

3. Given $3x - 2 + 24 = 31$, to find x. *Ans.* $x = 3$.

4. Given $x + 18 = 3x - 5$, to find x. *Ans.* $x = 11\frac{1}{2}$.

5. Given $6 - 2x + 10 = 20 - 3x - 2$, to find x. *Ans.* $x = 2$.

6. Given $x + \frac{1}{2}x + \frac{1}{3}x = 11$, to find x. *Ans.* $x = 6$.

7. Given $2x - \frac{1}{2}x + 1 = 5x - 2$, to find x. *Ans.* $x = \frac{5}{7}$.

Solve the following equations : —

8. $3ax + \frac{a}{2} - 3 = bx - a$. *Ans.* $x = \frac{6 - 3a}{6a - 2b}$.

9. $\frac{x - 3}{2} + \frac{x}{3} = 20 - \frac{x - 19}{2}$. *Ans.* $x = 23\frac{1}{4}$.

10. $\frac{x + 3}{2} + \frac{x}{3} = 4 - \frac{x - 5}{4}$. *Ans.* $x = 3\frac{6}{13}$.

11. $\frac{x}{4} - \frac{3x}{2} + x = \frac{4x}{8} - 3.$ *Ans.* $x = 4.$

12. $\frac{3ax}{c} - \frac{2bx}{d} - 4 = f.$ *Ans.* $x = \frac{cdf + 4cd}{3ad - 2bc}.$

13. $\frac{x - a}{3} - \frac{2x - 3b}{5} - \frac{a - x}{2} = 10a + 11b.$

Ans. $x = 25a + 24b.$

14. $\frac{x}{12} - \frac{8 - x}{8} - \frac{5 + x}{4} + \frac{11}{4} = 0.$ *Ans.* $x = 12.$

15. $\frac{a + c}{a + x} + \frac{a - c}{a - x} = \frac{2b^2}{a^2 - x^2}.$ *Ans.* $x = \frac{a^2 - b^2}{c}.$

16. $\frac{8ax - b}{7} - \frac{3b - c}{2} = 4 - b.$ *Ans.* $x = \frac{56 + 9b - 7c}{16a}.$

17. $\frac{x}{5} - \frac{x - 2}{3} + \frac{x}{2} = \frac{13}{3}.$ *Ans.* $x = 10.$

In the following equations, what is the numerical value of x, when $a = 1$, $b = 2$, $c = 3$, $d = 4$, and $f = 6$?

18. $\frac{x}{a} - \frac{x}{b} + \frac{x}{c} - \frac{x}{d} = f.$ *Ans.* $x = \frac{abcdf}{bcd - acd + abd - abc} = 10\frac{2}{7}.$

19. $\frac{x}{7} - \frac{8x}{9} - \frac{x - 3}{5} = -12\frac{29}{45}.$ *Ans.* $x = 14.$

20. $x - \frac{3x - 5}{13} + \frac{4x - 2}{11} = x + 1.$ *Ans.* $x = 6.$

21. $x + \frac{x}{4} + \frac{x}{5} - \frac{x}{6} = 2x - 43.$ *Ans.* $x = 60.$

22. $2x - \frac{4x - 2}{5} = \frac{3x - 1}{2}.$ *Ans.* $x = 3.$

23. $3x + \frac{bx - d}{3} = x + a.$ *Ans.* $x = \frac{3a + d}{6 + b} = \frac{7}{8}.$

24. $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}.$ *Ans.* $x=\frac{3b}{3a-2b}=-6.$

25. $\frac{4x}{5-x}-\frac{20-4x}{x}=\frac{15}{x}.$ *Ans.* $x=3\frac{2}{11}.$

26. $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}.$ *Ans.* $x=72.$

27. $\frac{(a+b)(b-x)}{a-b}-3a=\frac{4ab-b^2}{a+b}-2x+\frac{a^2-bx}{b}.$

Ans. $x=\frac{a(a^3+3a^2b+2ab^2-10b^3)}{b(2a^2-2ab-4b^2)}=\frac{65}{36}.$

Problems involving Equations of the First Degree.

109. A **problem** is a question proposed, requiring a solution.

The **solution** of a problem is the operation of finding a quantity, or quantities, that will satisfy the given conditions.

The solution of a problem consists of two parts, — the **statement** and the **solution.**

The **statement** consists in expressing algebraically the relation between the known and the required quantities.

The **solution** consists in finding the values of the unknown quantities in terms of those which are known.

The statement is made by representing the unknown quantities of the problem by some of the final letters of the alphabet, and then operating upon these so as to comply with the conditions of the problem. The method of stating problems is best learned by practical examples.

(1) What number is that to which if 5 be added the sum will be equal to 9?

Let $x=$ the number.

Then $x+5=9.$

This is the *statement* of the problem.

To find the value of x, transpose 5 to the second member.

Then $$x = 9 - 5 = 4.$$

This is the *solution* of the equation.

VERIFICATION. $$4 + 5 = 9.$$

(2) Find a number such that the sum of one half, one third, and one fourth of it, augmented by 45, shall be equal to 448.

Let $x =$ the required number.

Then $\frac{x}{2} =$ one half of it,

$\frac{x}{3} =$ one third of it,

$\frac{x}{4} =$ one fourth of it,

and, by the conditions,

$$\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + 45 = 448.$$

This is the statement of the problem.

Clearing of fractions,

$$6x + 4x + 3x + 540 = 5376.$$

Transposing, and collecting the unknown terms,

$$13x = 4836.$$

Hence $$x = \frac{4836}{13} = 372.$$

VERIFICATION.

$$\frac{372}{2} + \frac{372}{3} + \frac{372}{4} + 45 = 186 + 124 + 93 + 45 = 448.$$

(3) What number is that whose third part exceeds its fourth by 16?

Let $x =$ the required number.

Then $\frac{1}{3}x =$ the third part,

$\frac{1}{4}x =$ the fourth part,

and, by the conditions of the problem,

$$\frac{1}{3}x - \frac{1}{4}x = 16.$$

This is the statement.

Clearing of fractions, $4x - 3x = 192.$

Hence $x = 192.$

VERIFICATION. $\frac{192}{3} - \frac{192}{4} = 64 - 48 = 16.$

(4) Divide \$1000 between A, B, and C so that A shall have \$72 more than B, and C \$100 more than A.

Let $x =$ the number of dollars which B received.

Then $x =$ B's number,

$x + 72 =$ A's number,

$x + 172 =$ C's number;

and their sum, $3x + 244 = 1000$, the number of dollars.

This is the statement.

Transposing, $3x = 1000 - 244 = 756.$

$$x = \frac{756}{3} = 252 = \text{B's share.}$$

Hence $x + 72 = 252 + 72 = 324 =$ A's share.

$x + 172 = 252 + 172 = 424 =$ C's share.

VERIFICATION. $252 + 324 + 424 = 1000.$

(5) Out of a cask of wine which had leaked away a third part, 21 gallons were afterwards drawn, and the cask, being then gauged, was found to be half full. How much did it hold?

Let $x =$ the number of gallons.

Then $\frac{x}{3} =$ the number that had leaked away,

and $\frac{x}{3} + 21 =$ what had leaked and been drawn.

Hence $\frac{x}{3} + 21 = \frac{x}{2}.$

This is the statement.

Clearing of fractions,

$$2x + 126 = 3x,$$

and $$-x = -126;$$

and by changing the signs of both members, which does not destroy their equality (since it is equivalent to multiplying both members by -1), we have

$$x = 126.$$

VERIFICATION. $\frac{126}{3} + 21 = 42 + 21 = 63 = \frac{126}{2}.$

(6) A fish was caught whose tail weighed 9 pounds. His head weighed as much as his tail and half his body, and his body weighed as much as his head and tail together. What was the weight of the fish?

Let $2x =$ the weight of the body in pounds.

Then $9 + x =$ weight of the head;

and, since the body weighed as much as both head and tail,

$$2x = 9 + 9 + x,$$

which is the statement.

Then $2x - x = 18$, and $x = 18$.

Hence we have $2x = 36$ lbs. = weight of the body,

$9 + x = 27$ lbs. = weight of the head,

9 lbs. = weight of the tail.

Hence 72 lbs. = weight of the fish.

(7) The sum of two numbers is 67, and their difference 19. What are the two numbers?

First Method.

Let $x =$ the less number.

Then $x + 19 =$ the greater,

and $2x + 19 = 67.$

This is the statement.

24. $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}.$ *Ans.* $x=\frac{3b}{3a-2b}=-6.$

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Ans. $x=\frac{a(a^3+3a^2b+2ab^2-10b^3)}{b(2a^2-2ab-4b^2)}=\frac{65}{36}.$

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Clearing of fractions, $4x - 3x = 192.$

Hence $x = 192.$

VERIFICATION. $\frac{192}{3} - \frac{192}{4} = 64 - 48 = 16.$

(4) Divide \$1000 between A, B, and C so that A shall have \$72 more than B, and C \$100 more than A.

Let $x =$ the number of dollars which B received.

Then
$$\begin{aligned} x &= \text{B's number,} \\ x + 72 &= \text{A's number,} \\ x + 172 &= \text{C's number;} \end{aligned}$$

and their sum, $3x + 244 = 1000$, the number of dollars.

This is the statement.

Transposing, $3x = 1000 - 244 = 756.$

$$x = \frac{756}{3} = 252 = \text{B's share.}$$

Hence
$$\begin{aligned} x + 72 &= 252 + 72 = 324 = \text{A's share.} \\ x + 172 &= 252 + 172 = 424 = \text{C's share.} \end{aligned}$$

VERIFICATION. $252 + 324 + 424 = 1000.$

(5) Out of a cask of wine which had leaked away a third part, 21 gallons were afterwards drawn, and the cask, being then gauged, was found to be half full. How much did it hold?

Let $x =$ the number of gallons.

Then $\frac{x}{3} =$ the number that had leaked away,

and $\frac{x}{3} + 21 =$ what had leaked and been drawn.

Hence $\frac{x}{3} + 21 = \frac{x}{2}.$

This is the statement.

Clearing of fractions,

$$2x + 126 = 3x,$$

and $$-x = -126;$$

and by changing the signs of both members, which does not destroy their equality (since it is equivalent to multiplying both members by -1), we have

$$x = 126.$$

VERIFICATION. $\frac{126}{3} + 21 = 42 + 21 = 63 = \frac{126}{2}.$

(6) A fish was caught whose tail weighed 9 pounds. His head weighed as much as his tail and half his body, and his body weighed as much as his head and tail together. What was the weight of the fish?

Let $2x =$ the weight of the body in pounds.

Then $9 + x =$ weight of the head;

and, since the body weighed as much as both head and tail,

$$2x = 9 + 9 + x,$$

which is the statement.

Then $2x - x = 18$, and $x = 18$.

Hence we have $2x = 36$ lbs. = weight of the body,

$9 + x = 27$ lbs. = weight of the head,

9 lbs. = weight of the tail.

Hence 72 lbs. = weight of the fish.

(7) The sum of two numbers is 67, and their difference 19. What are the two numbers?

First Method.

Let $x =$ the less number.

Then $x + 19 =$ the greater,

and $2x + 19 = 67$.

This is the statement.

Transposing, $2x = 67 - 19 = 48.$

Hence $x = \frac{48}{2} = 24$, and $x + 19 = 43.$

VERIFICATION. $43 + 24 = 67$, and $43 - 24 = 19.$

Second Method.

Let $x =$ the greater number.

Then $x - 19 =$ the less,

and $2x - 19 = 67$; whence $2x = 67 + 19.$

$$\therefore\ x = \frac{86}{2} = 43;$$

consequently $x - 19 = 43 - 19 = 24.$

As a general solution of this problem, take the following: —

The sum of two numbers is s. Their difference is d. What are the two numbers?

Let $x =$ the *less* number.

Then $x + d =$ the greater,

and $2x + d = s$, their sum.

Whence $x = \frac{s - d}{2} = \frac{s}{2} - \frac{d}{2};$

and consequently $x + d = \frac{s}{2} - \frac{d}{2} + d = \frac{s}{2} + \frac{d}{2}.$

As these two results are not dependent on particular values attributed to s or d, it follows that

The greater of two numbers is equal to half their sum, plus half their difference.

The less is equal to half their sum, minus half their difference.

Thus, if the sum of two numbers is 32, and their difference 16, the greater is

$$\frac{32}{2} + \frac{16}{2} = 16 + 8 = 24;$$

the less, $\frac{32}{2} - \frac{16}{2} = 16 - 8 = 8.$

VERIFICATION. $24 + 8 = 32$, and $24 - 8 = 16.$

(8) A person engaged a workman for 48 days. For each day that he labored he received 24 cents, and for each day that he was idle he paid 12 cents for his board. At the end of the 48 days the account was settled, when the laborer received 504 cents. Required the number of working days, and the number of days he was idle.

If the number of working days and the number of idle days were known, and the first multiplied by 24 and the second by 12, the difference of these products would be 504. Let us indicate these operations by means of algebraic signs.

Let $x =$ the number of working days.

Then $48 - x =$ the number of idle days,

$24 \times x =$ the amount earned,

and $12(48 - x) =$ the amount paid for board.

Then $24x - 12(48 - x) = 504$,

what was received, which is the statement.

Then, performing the operations indicated,

$$24x - 576 + 12x = 504,$$

or $36x = 504 + 576 = 1080$,

and $x = \frac{1080}{36} = 30$, the number of working days;

whence $48 - 30 = 18$, the number of idle days.

VERIFICATION.

30 days' labor at 24 cents $= 30 \times 24 = 720$ cents.
18 days' board at 12 cents $= 18 \times 12 = 216$ cents.

Difference, or amount received $= 504$ cents.

This problem may be made general by denoting the whole number of working and idle days by n; the amount received for each day's work, by a; the amount paid for board for each idle day, by b; and what was due the laborer, or the balance of the account, by c.

As before, let the number of working days be denoted by x. The number of idle days will then be denoted by $n - x$.

Hence what is earned will be expressed by ax; and the sum to be deducted on account of board, by $b(n-x)$.

The statement of the problem, therefore, is,

$$ax - b(n-x) = c.$$

Performing the indicated operations,

$$ax - bn + bx = c, \text{ or } (a+b)x = c + bn.$$

Whence $$x = \frac{c+bn}{a+b} = \text{number of working days};$$

and $$n - x = n - \frac{c+bn}{a+b} = \frac{an+bn-c-bn}{a+b},$$

or $$n - x = \frac{an-c}{a+b} = \text{number of idle days}.$$

Let us suppose $n = 48$, $a = 24$, $b = 12$, and $c = 504$. These numbers will give for x the same value as before found.

(9) A person, dying, leaves half of his property to his wife, one sixth to each of two daughters, one twelfth to a servant, and the remaining \$600 to the poor. What was the amount of the property?

Let $x =$ the amount in dollars.

Then $\frac{x}{2} =$ what he left to his wife,

$\frac{x}{6} =$ what he left to one daughter,

and $\frac{2x}{6} = \frac{x}{3}$, what he left to both daughters;

also $\frac{x}{12} =$ what he left to his servant,

and \$600 = what he left to the poor.

Then, by the conditions,

$$\frac{x}{2} + \frac{x}{3} + \frac{x}{12} + 600 = x, \text{ the amount of the property,}$$

which gives $x = \$7200$.

(10) A and B play together at cards. A sits down with \$84, and B with \$48. Each loses and wins in turn, when it appears that A has five times as much as B. How much did A win?

Let x denote the number of dollars A won.

Then A rose with $84 + x$ dollars,

and B rose with $48 - x$ dollars.

But by the conditions we have

$$84 + x = 5(48 - x).$$

Hence $$84 + x = 240 - 5x,$$

and $$6x = 156.$$

Consequently $x = 26$, or A won \$26.

VERIFICATION.

$$84 + 26 = 110,\ 48 - 26 = 22,\ 110 = 5(22) = 110.$$

(11) A can do a piece of work alone in 10 days, and B in 13 days. In what time can they do it if they work together?

Denote the time by x, and the work to be done by 1. Then

In 1 day A can do $\frac{1}{10}$ of the work,

and B can do $\frac{1}{13}$ of the work;

and In x days A can do $\frac{x}{10}$ of the work,

and B can do $\frac{x}{13}$ of the work.

Hence, by the conditions,

$$\frac{x}{10} + \frac{x}{13} = 1, \text{ which gives } 13x + 10x = 130.$$

Hence $$23x = 130,\ x = \frac{130}{23} = 5\tfrac{15}{23} \text{ days}.$$

(12) A fox, pursued by a hound, has a start of 60 of his own leaps. Three leaps of the hound are equivalent to 7 of the fox; but while the hound makes 6 leaps, the fox makes 9. How many leaps must the hound make to overtake the fox?

There is some difficulty in this problem, arising from the different units which enter into it.

Since 3 leaps of the hound are equal to 7 leaps of the fox, 1 leap of the hound is equal to $\frac{7}{3}$ fox leaps.

Since, while the hound makes 6 leaps, the fox makes 9, while the hound makes 1 leap, the fox will make $\frac{9}{6}$, or $\frac{3}{2}$, leaps.

Let x denote the *number* of leaps which the hound makes before he overtakes the fox, and let 1 fox leap denote the *unit of distance.*

Since 1 leap of the hound is equal to $\frac{7}{3}$ of a fox leap, x leaps will be equal to $\frac{7}{3}x$ fox leaps; and this will denote the distance passed over by the hound in fox leaps.

Since, while the hound makes 1 leap, the fox makes $\frac{3}{2}$ leaps, while the hound makes x leaps, the fox makes $\frac{3}{2}x$ leaps; and this added to 60, his distance ahead, will give $\frac{3}{2}x + 60$ for the whole distance passed over by the fox.

Hence, from the conditions,

$$\frac{7}{3}x = \frac{3}{2}x + 60.$$

Whence

$$14x = 9x + 360,$$

$$x = 72.$$

The hound, therefore, makes 72 leaps before overtaking the fox. In the same time, the fox makes $72 \times \frac{3}{2} = 108$ leaps.

VERIFICATION. $108 + 60 = 168$, whole number of fox leaps,

$$72 \times \frac{7}{3} = 168.$$

Exercises.

1. A father leaves his property, amounting to \$2520, to four sons, A, B, C, and D. C is to have \$360; B, as much as C and D together; and A, twice as much as B, less \$1000. How much do A, B, and D receive?

Ans. A, \$760; B, \$880; D, \$520.

2. An estate of $7500 is to be divided among a widow, two sons, and three daughters, so that each son shall receive twice as much as each daughter, and the widow herself $500 more than all the children. What was her share, and what the share of each child?

Ans. Widow's share, $4000. Each son's, $1000. Each daughter's, $500.

3. A company of 180 persons consists of men, women, and children. The men are 8 more in number than the women, and the children 20 more than the men and women together. How many of each sort in the company?

Ans. 44 men, 36 women, 100 children.

4. A father divides $2000 among five sons, so that each elder should receive $40 more than his next younger brother. What is the share of the youngest? *Ans.* $320.

5. A purse of $2850 is to be divided among three persons, A, B, and C. A's share is to be to B's as 6 to 11, and C is to have $300 more than A and B together. What is each one's share? *Ans.* A's, $450; B's, $825; C's, $1575.

6. Two pedestrians start from the same point and travel in the same direction. The first steps twice as far as the second; but the second makes 5 steps while the first makes but 1. At the end of a certain time they are 300 feet apart. Now, allowing each of the longer paces to be 3 feet, how far will each have traveled? *Ans.* 1st, 200 feet; 2d, 500 feet.

7. Two carpenters, 24 journeymen, and 8 apprentices received at the end of a certain time $144. The carpenters received $1 per day; each journeyman, half a dollar; and each apprentice, 25 cents. How many days were they employed? *Ans.* 9 days.

8. A capitalist receives a yearly income of $2940. Four fifths of his money bears an interest of 4 per cent, and the remainder 5 per cent. How much has he at interest?

Ans. $70,000.

9. A cistern containing 60 gallons of water has three unequal cocks for discharging it. The largest will empty it in one hour; the second, in two hours; and the third, in three. In what time will the cistern be emptied if they all run together? *Ans.* $32\frac{8}{11}$ minutes.

10. In a certain orchard one half are apple trees; one fourth, peach trees; one sixth, plum trees. There are also 120 cherry trees and 80 pear trees. How many trees in the orchard? *Ans.* 2400.

11. A farmer, being asked how many sheep he had, answered that he had them in five fields. In the first he had $\frac{1}{4}$; in the second, $\frac{1}{6}$; in the third, $\frac{1}{8}$; in the fourth, $\frac{1}{12}$; and in the fifth, 450. How many had he? *Ans.* 1200.

12. My horse and saddle together are worth $132, and the horse is worth ten times as much as the saddle. What is the value of the horse? *Ans.* $120.

13. The rent of an estate is this year 8 per cent greater than it was last. This year it is $1890. What was it last year? *Ans.* $1750.

14. What number is that from which, if 5 be subtracted, $\frac{2}{3}$ of the remainder will be 40? *Ans.* 65.

15. A post is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and 10 feet above the water. What is the whole length of the post?

Ans. 24 feet.

16. After paying $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left in my purse. How many guineas were in it at first?

Ans. 120.

17. A person was desirous of giving 3 pence apiece to some beggars, but found he had not money enough in his pocket by 8 pence. He therefore gave them each 2 pence, and had 3 pence remaining. Required the number of beggars. *Ans.* 11.

18. A person, in play, lost $\frac{1}{4}$ of his money, and then won 3 shillings, after which he lost $\frac{1}{3}$ of what he then had, and, this done, found that he had but 12 shillings remaining. What had he at first? *Ans.* 20 shillings.

19. Two persons, A and B, lay out equal sums of money in trade. A gains $126, and B loses $87, and A's money is then double B's. What did each lay out? *Ans.* $300.

20. A person goes to a tavern with a certain sum of money in his pocket, where he spends 2 shillings. He then borrows as much money as he had left, and, going to another tavern, he there spends 2 shillings also. Then borrowing again as much money as was left, he went to a third tavern, where likewise he spent 2 shillings, and borrowed as much as he had left; and, again spending 2 shillings at a fourth tavern, he then had nothing remaining. What had he at first?

Ans. 3s. 9d.

21. A tailor cut 19 yards from each of three equal pieces of cloth, and 17 yards from another of the same length, and found that the four remnants were together equal to 142 yards. How many yards in each piece? *Ans.* 54.

22. A fortress is garrisoned by 2600 men, consisting of infantry, artillery, and cavalry. Now, there are nine times as many infantry, and three times as many artillery, as there are cavalry. How many are there of each corps?

Ans. 200 cavalry, 600 artillery, 1800 infantry.

23. All the journeyings of an individual amounted to 2970 miles. Of these, he traveled $3\frac{1}{2}$ times as many by water as

on horseback, and $2\frac{1}{4}$ times as many on foot as by water. How many miles did he travel in each way?

Ans. 240 miles, 840 miles, 1890 miles.

24. A sum of money was divided between two persons, A and B. A's share was to B's in the proportion of 5 to 3, and exceeded five ninths of the entire sum by $50. What was the share of each? *Ans.* A's share, $450; B's, $270.

25. Divide a number a into three such parts that the second shall be n times the first, and the third m times as great as the first.

Ans. 1st, $\frac{a}{1+m+n}$; 2d, $\frac{na}{1+m+n}$; 3d, $\frac{ma}{1+m+n}$.

26. A father directs that $1170 shall be divided among his three sons in proportion to their ages. The oldest is twice as old as the youngest, and the second is one third older than the youngest. How much was each to receive?

Ans. Youngest, $270; second, $360; oldest, $540.

27. Three regiments are to furnish 594 men, and each to furnish in proportion to its strength. Now, the strength of the first is to the strength of the second as 3 to 5; and that of the second to that of the third as 8 to 7. How many must each furnish?

Ans. 1st, 144 men; 2d, 240 men; 3d, 210 men.

28. Five heirs, A, B, C, D, and E, are to divide an inheritance of $5600. B is to receive twice as much as A, and $200 more; C, three times as much as A, less $400; D, the half of what B and C receive together, and $150 more; and E, the fourth part of what the four others get, plus $475. How much did each receive?

Ans. A, $500; B, $1200; C, $1100; D, $1300; E, $1500.

29. A person has four casks, the second of which being filled from the first, leaves the first four sevenths full; the

third being filled from the second, leaves it one fourth full; and when the third is emptied into the fourth, it is found to fill only nine sixteenths of it. But the first will fill the third and fourth, and leave 15 quarts remaining. How many gallons does each hold?

Ans. 1st, 35 gals.; 2d, 15 gals.; 3d, 11¼ gals.; 4th, 20 gals.

30. A courier, having started from a place, is pursued by a second after the lapse of 10 days. The first travels 4 miles a day; the other, 9. How many days before the second will overtake the first? *Ans.* 8 days.

31. A courier goes 31½ miles every five hours, and is followed by another after he had been gone eight hours. The second travels 22½ miles every three hours. How many hours before he will overtake the first? *Ans.* 42 hours.

32. Two places are 80 miles apart, and a person leaves one of them and travels towards the other at the rate of 3½ miles per hour. Eight hours after, a person departs from the second place, and travels at the rate of 5⅛ miles per hour. How long before they will be together? *Ans.* 6 hours.

Equations containing Two Unknown Quantities.

110. If we have a single equation, as

$$2x + 3y = 21,$$

containing two unknown quantities, x and y, we may find the value of one of them in terms of the other; as

$$x = \frac{21 - 3y}{2} \qquad (1)$$

Now, if the value of y is unknown, that of x will also be unknown. Hence, from a *single* equation containing two unknown quantities, the value of x cannot be determined.

If we have a second equation, as

$$5x + 4y = 35,$$

we may, as before, find the value of x in terms of y, giving

$$x = \frac{35 - 4y}{5} \tag{2}$$

Now, if the values of x and y are the same in Equations (1) and (2), the second members may be placed equal to each other, giving

$$\frac{21 - 3y}{2} = \frac{35 - 4y}{5},$$

or

$$105 - 15y = 70 - 8y,$$

from which we find $y = 5$.

Substituting this value for y in Equations (1) or (2), we find $x = 3$. Such equations are called **simultaneous equations.** Hence

111. Simultaneous equations are those in which the values of the unknown quantities are the same in both.

Elimination.

112. Elimination is the operation of combining two equations containing two unknown quantities, and deducing therefrom a single equation containing but one.

There are three principal methods of elimination, —

I. By addition or subtraction.
II. By substitution.
III. By comparison.

We shall consider these methods separately.

I. Elimination by Addition or Subtraction.

(1) Take the two equations

$$3x - 2y = 7,$$
$$8x + 2y = 48.$$

If we add these two equations, member to member, we obtain

$$11x = 55,$$

which gives, by dividing by 11,

$$x = 5;$$

and substituting this value in either of the given equations, we find

$$y = 4.$$

(2) Again, take the equations

$$8x + 2y = 48,$$
$$3x + 2y = 23.$$

If we subtract the 2d equation from the 1st, we obtain

$$5x = 25,$$

which gives, by dividing by 5,

$$x = 5;$$

and by substituting this value, we find

$$y = 4.$$

(3) Given the sum of two numbers equal to s, and their difference equal to d, to find the numbers.

Let $x =$ the greater, and y the less, number.

Then $x + y = s,$

and $x - y = d.$

By adding (§ 102, Axiom 1), $2x = s + d.$

By subtracting (§ 102, Axiom 2),

$$2y = s - d.$$

Each of these equations contains but one unknown quantity.

From the first we obtain $x = \frac{s + d}{2};$

and from the second, $y = \frac{s - d}{2}.$

NOTE. — These are the same values as were found in Problem (7), p. 114.

(4) A person engaged a workman for 48 days. For each day that he labored he was to receive 24 cents, and for each day that he was idle he was to pay 12 cents for his board. At the end of the 48 days the account was settled, when the laborer received 504 cents. Required the number of working days, and the number of days he was idle.

Let $x =$ the number of working days,

and $y =$ the number of idle days.

Then $24x =$ what he earned,

and $12y =$ what he paid for his board.

Then, by the conditions of the question, we have

$$x + y = 48,$$

and

$$24x - 12y = 504.$$

This is the statement of the problem.

It has already been shown (§ 102, Axiom 3) that the two members of an equation may be multiplied by the same number without destroying the equality. Let, then, the members of the first equation be multiplied by 24, the coefficient of x in the second. We shall then have

$$\begin{aligned} 24x + 24y &= 1152 \\ 24x - 12y &= 504 \end{aligned}$$

and, by subtracting,

$$36y = 648$$

$$\therefore\ y = \frac{648}{36} = 18.$$

Substituting this value of y in the equation

$$24x - 12y = 504,$$

we have

$$24x - 216 = 504,$$

which gives

$$24x = 504 + 216 = 720,$$

and

$$x = \frac{720}{24} = 30.$$

(See p. 115, where this problem is solved by means of one unknown quantity.)

113. In a similar manner, either unknown quantity may be eliminated. Hence the following rule: —

Prepare the equations so that the coefficients of the quantity to be eliminated shall be numerically equal.

If the signs are unlike, add the equations, member to member; if alike, subtract them, member from member.

Exercises.

Find the values of x and y, by addition or subtraction, in the following simultaneous equations: —

1. $\begin{Bmatrix} 3x - y = 3. \\ y + 2x = 7. \end{Bmatrix}$ *Ans.* $x = 2,\ y = 3.$

2. $\begin{Bmatrix} 4x - 7y = -22. \\ 5x + 2y = 37. \end{Bmatrix}$ *Ans.* $x = 5,\ y = 6.$

3. $\begin{Bmatrix} 2x + 6y = 42. \\ 8x - 6y = 3. \end{Bmatrix}$ *Ans.* $x = 4\frac{1}{2},\ y = 5\frac{1}{2}.$

4. $\begin{Bmatrix} 8x - 9y = 1. \\ 6x - 3y = 4x. \end{Bmatrix}$ *Ans.* $x = \frac{1}{2},\ y = \frac{1}{3}.$

5. $\begin{Bmatrix} 14x - 15y = 12. \\ 7x + 8y = 37. \end{Bmatrix}$ *Ans.* $x = 3,\ y = 2.$

6. $\begin{Bmatrix} \frac{1}{2}x + \frac{1}{3}y = 6. \\ \frac{1}{3}x + \frac{1}{2}y = 6\frac{1}{2}. \end{Bmatrix}$ *Ans.* $x = 6,\ y = 9.$

7. $\begin{Bmatrix} \frac{1}{7}x + \frac{1}{8}y = 4. \\ x - y = -2. \end{Bmatrix}$ *Ans.* $x = 14,\ y = 16.$

8. Says A to B, "You give me \$40 of your money, and I shall then have five times as much as you will have left." Now, they both had \$120. How much had each? *Ans.* \$60.

9. A father says to his son, "Twenty years ago my age was four times yours: now it is just double." What were their ages? *Ans.* Father's, 60 years; son's, 30 years.

10. A father divided his property between his two sons. At the end of the first year the elder had spent one quarter of his, and the younger had made \$1000, and their property was then equal. After this, the elder spent \$500, and the younger made \$2000, when it appeared that the younger had just double the elder. What had each from the father?

Ans. Elder, \$4000; younger, \$2000.

11. If John gives Charles 15 apples, they will have the same number; but if Charles gives 15 to John, John will have 15 times as many, wanting 10, as Charles will have left. How many has each? *Ans.* John, 50; Charles, 20.

12. Two clerks, A and B, have salaries which are together equal to \$900. A spends $\frac{1}{10}$ per year of what he receives, and B adds as much to his as A spends. At the end of the year they have equal sums. What was the salary of each?

Ans. A's, \$500; B's, \$400.

II. Elimination by Substitution.

114. Let us take the equations

$$5x + 7y = 43 \qquad (1)$$

$$11x + 9y = 69 \qquad (2)$$

Find the value of x in the first equation, which gives

$$x = \frac{43 - 7y}{5}.$$

Substitute this value of x in the second equation, and we have

$$11 \times \frac{43 - 7y}{5} + 9y = 69,$$

or

$$473 - 77y + 45y = 345,$$

or

$$-32y = -128.$$

Here x has been eliminated by *substitution.*

In a similar manner we can eliminate any unknown quantity. Hence the rule: —

Find from either equation the value of the unknown quantity to be eliminated.

Substitute this value for that quantity in the other equation.

NOTE. — This method of elimination is used to great advantage when the coefficient of either of the unknown quantities is 1.

Exercises.

Find by the last method the values of x and y in the following equations: —

1. $3x - y = 1$, and $3y - 2x = 4$. *Ans.* $x = 1$, $y = 2$.
2. $5y - 4x = -22$, and $3y + 4x = 38$. *Ans.* $x = 8$, $y = 2$.
3. $x + 8y = 18$, and $y - 3x = -29$. *Ans.* $x = 10$, $y = 1$.
4. $5x - y = 13$, and $8x + \frac{2}{9}y = 29$. *Ans.* $x = 3\frac{1}{2}$, $y = 4\frac{1}{2}$.
5. $10x - \frac{y}{5} = 69$, and $10y - \frac{x}{7} = 49$. *Ans.* $x = 7$, $y = 5$.
6. $x + \frac{1}{2}x - \frac{y}{5} = 10$, and $\frac{x}{8} + \frac{y}{10} = 2$. *Ans.* $x = 8$, $y = 10$.
7. $\frac{y}{7} - \frac{x}{3} + 5 = 2$, and $x + \frac{y}{5} = 17\frac{4}{5}$. *Ans.* $x = 15$, $y = 14$.

8. $\frac{y}{2}+\frac{x}{3}+3=6\frac{1}{6}$, and $\frac{y}{4}-\frac{x}{7}=\frac{1}{2}$. *Ans.* $x=3\frac{1}{2}$, $y=4$.

9. $\frac{y}{8}-\frac{x}{4}+6=5$, and $\frac{x}{12}-\frac{y}{16}=0$. *Ans.* $x=12$, $y=16$.

10. $\frac{y}{7}-\frac{3x}{2}-1=-9$, and $5x-\frac{7y}{49}=29$.

Ans. $x=6$, $y=7$.

11. Two misers, A and B, sit down to count over their money. Together they have $20,000, and B has three times as much as A. How much has each?

Ans. A, $5000; B, $15,000.

12. A person has two purses. If he puts $7 into the first, the whole is worth three times as much as the second purse; but if he puts $7 into the second, the whole is worth five times as much as the first. What is the value of each purse?

Ans. 1st, $2; 2d, $3.

13. Two numbers have the following relations: if the first be multiplied by 6, the product will be equal to the second multiplied by 5; and 1 subtracted from the first leaves the same remainder as 2 subtracted from the second. What are the numbers? *Ans.* 5 and 6.

14. Find two numbers with the following relations: the first increased by 2 is $3\frac{1}{4}$ times as great as the second; and the second increased by 4 gives a number equal to half the first. *Ans.* 24 and 8.

15. A father says to his son, "Twelve years ago I was twice as old as you are now; four times your age at that time, plus twelve years, will express my age twelve years hence." What were their ages?

Ans. Father, 72 years; son, 30 years.

III. Elimination by Comparison.

115. Take again the equations,

$$5x + 7y = 43,$$
$$11x + 9y = 69.$$

Finding the value of x from the first equation, we have

$$x = \frac{43 - 7y}{5};$$

and finding the value of x from the second, we obtain

$$x = \frac{69 - 9y}{11}.$$

Let these two values of x be placed equal to each other, and we have

$$\frac{43 - 7y}{5} = \frac{69 - 9y}{11},$$

or $$473 - 77y = 345 - 45y,$$

or $$-32y = -128.$$

Hence $$y = 4,$$

and $$x = \frac{69 - 36}{11} = 3.$$

This method of elimination is called the method by **comparison,** for which we have the following rule:—

Find from each equation the value of the same unknown quantity to be eliminated.

Place these values equal to each other.

Exercises.

Find by the last rule the values of x and y from the following equations:—

1. $3x + \frac{y}{5} + 6 = 42$, and $y - \frac{x}{22} = 14\frac{1}{2}$.

Ans. $x = 11$, $y = 15$.

If we have a second equation, as

$$5x + 4y = 35,$$

we may, as before, find the value of x in terms of y, giving

$$x = \frac{35 - 4y}{5} \qquad (2)$$

Now, if the values of x and y are the same in Equations (1) and (2), the second members may be placed equal to each other, giving

$$\frac{21 - 3y}{2} = \frac{35 - 4y}{5},$$

or $$105 - 15y = 70 - 8y,$$

from which we find $y = 5$.

Substituting this value for y in Equations (1) or (2), we find $x = 3$. Such equations are called **simultaneous equations.** Hence

111. Simultaneous equations are those in which the values of the unknown quantities are the same in both.

Elimination.

112. Elimination is the operation of combining two equations containing two unknown quantities, and deducing therefrom a single equation containing but one.

There are three principal methods of elimination, —

I. By addition or subtraction.
II. By substitution.
III. By comparison.

We shall consider these methods separately.

I. Elimination by Addition or Subtraction.

(1) Take the two equations

$$3x - 2y = 7,$$
$$8x + 2y = 48.$$

If we add these two equations, member to member, we obtain

$$11x = 55,$$

which gives, by dividing by 11,

$$x = 5;$$

and substituting this value in either of the given equations, we find

$$y = 4.$$

(2) Again, take the equations

$$8x + 2y = 48,$$
$$3x + 2y = 23.$$

If we subtract the 2d equation from the 1st, we obtain

$$5x = 25,$$

which gives, by dividing by 5,

$$x = 5;$$

and by substituting this value, we find

$$y = 4.$$

(3) Given the sum of two numbers equal to s, and their difference equal to d, to find the numbers.

Let $x =$ the greater, and y the less, number.

Then $x + y = s,$

and $x - y = d.$

By adding (§ 102, Axiom 1), $2x = s + d.$

By subtracting (§ 102, Axiom 2),

$$2y = s - d.$$

Each of these equations contains but one unknown quantity.

From the first we obtain $x = \frac{s + d}{2};$

and from the second, $y = \frac{s - d}{2}.$

NOTE. — These are the same values as were found in Problem (7), p. 114.

(4) A person engaged a workman for 48 days. For each day that he labored he was to receive 24 cents, and for each day that he was idle he was to pay 12 cents for his board. At the end of the 48 days the account was settled, when the laborer received 504 cents. Required the number of working days, and the number of days he was idle.

Let $x =$ the number of working days,

and $y =$ the number of idle days.

Then $24x =$ what he earned,

and $12y =$ what he paid for his board.

Then, by the conditions of the question, we have

$$x + y = 48,$$

and

$$24x - 12y = 504.$$

This is the statement of the problem.

It has already been shown (§ 102, Axiom 3) that the two members of an equation may be multiplied by the same number without destroying the equality. Let, then, the members of the first equation be multiplied by 24, the coefficient of x in the second. We shall then have

$$24x + 24y = 1152$$
$$24x - 12y = 504$$

and, by subtracting,

$$36y = 648$$

$$\therefore y = \frac{648}{36} = 18.$$

Substituting this value of y in the equation

$$24x - 12y = 504,$$

we have

$$24x - 216 = 504,$$

which gives

$$24x = 504 + 216 = 720,$$

and

$$x = \frac{720}{24} = 30.$$

(See p. 115, where this problem is solved by means of one unknown quantity.)

113. In a similar manner, either unknown quantity may be eliminated. Hence the following rule: —

Prepare the equations so that the coefficients of the quantity to be eliminated shall be numerically equal.

If the signs are unlike, add the equations, member to member; if alike, subtract them, member from member.

Exercises.

Find the values of x and y, by addition or subtraction, in the following simultaneous equations: —

1. $\left\{\begin{array}{l} 3x - y = 3. \\ y + 2x = 7. \end{array}\right\}$ *Ans.* $x = 2,\ y = 3.$

2. $\left\{\begin{array}{l} 4x - 7y = -22. \\ 5x + 2y = 37. \end{array}\right\}$ *Ans.* $x = 5,\ y = 6.$

3. $\left\{\begin{array}{l} 2x + 6y = 42. \\ 8x - 6y = 3. \end{array}\right\}$ *Ans.* $x = 4\frac{1}{2},\ y = 5\frac{1}{2}.$

4. $\left\{\begin{array}{l} 8x - 9y = 1. \\ 6x - 3y = 4x. \end{array}\right\}$ *Ans.* $x = \frac{1}{2},\ y = \frac{1}{3}.$

5. $\left\{\begin{array}{l} 14x - 15y = 12. \\ 7x + 8y = 37. \end{array}\right\}$ *Ans.* $x = 3,\ y = 2.$

6. $\left\{\begin{array}{l} \frac{1}{2}x + \frac{1}{3}y = 6. \\ \frac{1}{3}x + \frac{1}{2}y = 6\frac{1}{2}. \end{array}\right\}$ *Ans.* $x = 6,\ y = 9.$

7. $\left\{\begin{array}{l} \frac{1}{7}x + \frac{1}{8}y = 4. \\ x - y = -2. \end{array}\right\}$ *Ans.* $x = 14,\ y = 16.$

8. Says A to B, "You give me \$40 of your money, and I shall then have five times as much as you will have left." Now, they both had \$120. How much had each? *Ans.* \$60.

9. A father says to his son, "Twenty years ago my age was four times yours: now it is just double." What were their ages? *Ans.* Father's, 60 years; son's, 30 years.

10. A father divided his property between his two sons. At the end of the first year the elder had spent one quarter of his, and the younger had made \$1000, and their property was then equal. After this, the elder spent \$500, and the younger made \$2000, when it appeared that the younger had just double the elder. What had each from the father?
Ans. Elder, \$4000; younger, \$2000.

11. If John gives Charles 15 apples, they will have the same number; but if Charles gives 15 to John, John will have 15 times as many, wanting 10, as Charles will have left. How many has each? *Ans.* John, 50; Charles, 20.

12. Two clerks, A and B, have salaries which are together equal to \$900. A spends $\frac{1}{10}$ per year of what he receives, and B adds as much to his as A spends. At the end of the year they have equal sums. What was the salary of each?
Ans. A's, \$500; B's, \$400.

II. Elimination by Substitution.

114. Let us take the equations

$$5x + 7y = 43 \qquad (1)$$

$$11x + 9y = 69 \qquad (2)$$

Find the value of x in the first equation, which gives

$$x = \frac{43 - 7y}{5}.$$

Substitute this value of x in the second equation, and we have

$$11 \times \frac{43 - 7y}{5} + 9y = 69,$$

or $$473 - 77y + 45y = 345,$$

or $$-32y = -128.$$

Here x has been eliminated by *substitution.*

In a similar manner we can eliminate any unknown quantity. Hence the rule: —

Find from either equation the value of the unknown quantity to be eliminated.

Substitute this value for that quantity in the other equation.

NOTE. — This method of elimination is used to great advantage when the coefficient of either of the unknown quantities is 1.

Exercises.

Find by the last method the values of x and y in the following equations: —

1. $3x - y = 1$, and $3y - 2x = 4$. *Ans.* $x = 1$, $y = 2$.
2. $5y - 4x = -22$, and $3y + 4x = 38$. *Ans.* $x = 8$, $y = 2$.
3. $x + 8y = 18$, and $y - 3x = -29$. *Ans.* $x = 10$, $y = 1$.
4. $5x - y = 13$, and $8x + \frac{2}{9}y = 29$. *Ans.* $x = 3\frac{1}{2}$, $y = 4\frac{1}{2}$.
5. $10x - \frac{y}{5} = 69$, and $10y - \frac{x}{7} = 49$. *Ans.* $x = 7$, $y = 5$.
6. $x + \frac{1}{2}x - \frac{y}{5} = 10$, and $\frac{x}{8} + \frac{y}{10} = 2$. *Ans.* $x = 8$, $y = 10$.
7. $\frac{y}{7} - \frac{x}{3} + 5 = 2$, and $x + \frac{y}{5} = 17\frac{4}{5}$. *Ans.* $x = 15$, $y = 14$.

8. $\frac{y}{2}+\frac{x}{3}+3=6\frac{1}{6}$, and $\frac{y}{4}-\frac{x}{7}=\frac{1}{2}$. *Ans.* $x=3\frac{1}{2}$, $y=4$.

9. $\frac{y}{8}-\frac{x}{4}+6=5$, and $\frac{x}{12}-\frac{y}{16}=0$. *Ans.* $x=12$, $y=16$.

10. $\frac{y}{7}-\frac{3x}{2}-1=-9$, and $5x-\frac{7y}{49}=29$.

Ans. $x=6$, $y=7$.

11. Two misers, A and B, sit down to count over their money. Together they have $20,000, and B has three times as much as A. How much has each?

Ans. A, $5000; B, $15,000.

12. A person has two purses. If he puts $7 into the first, the whole is worth three times as much as the second purse; but if he puts $7 into the second, the whole is worth five times as much as the first. What is the value of each purse?

Ans. 1st, $2; 2d, $3.

13. Two numbers have the following relations: if the first be multiplied by 6, the product will be equal to the second multiplied by 5; and 1 subtracted from the first leaves the same remainder as 2 subtracted from the second. What are the numbers? *Ans.* 5 and 6.

14. Find two numbers with the following relations: the first increased by 2 is $3\frac{1}{4}$ times as great as the second; and the second increased by 4 gives a number equal to half the first. *Ans.* 24 and 8.

15. A father says to his son, "Twelve years ago I was twice as old as you are now; four times your age at that time, plus twelve years, will express my age twelve years hence." What were their ages?

Ans. Father, 72 years; son, 30 years.

III. Elimination by Comparison.

115. Take again the equations,

$$5x + 7y = 43,$$
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Finding the value of x from the first equation, we have

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and finding the value of x from the second, we obtain

$$x = \frac{69 - 9y}{11}.$$

Let these two values of x be placed equal to each other, and we have

$$\frac{43 - 7y}{5} = \frac{69 - 9y}{11},$$

or $$473 - 77y = 345 - 45y,$$

or $$-32y = -128.$$

Hence $$y = 4,$$

and $$x = \frac{69 - 36}{11} = 3.$$

This method of elimination is called the method by **comparison,** for which we have the following rule: —

Find from each equation the value of the same unknown quantity to be eliminated.

Place these values equal to each other.

Exercises.

Find by the last rule the values of x and y from the following equations: —

1. $3x + \frac{y}{5} + 6 = 42$, and $y - \frac{x}{22} = 14\frac{1}{2}$.

Ans. $x = 11$, $y = 15$.

2. $\frac{y}{4}-\frac{x}{7}+5=6$, and $\frac{y}{5}+4=\frac{x}{14}+6$.

Ans. $x=28$, $y=20$.

3. $\frac{y}{10}-\frac{x}{4}+\frac{22}{8}=1$, and $3y-x=6$. *Ans.* $x=9$, $y=5$.

4. $y-3=\frac{1}{2}x+5$, and $\frac{x+y}{2}=y-3\frac{1}{2}$. *Ans.* $x=2$, $y=9$.

5. $\frac{y-x}{3}+\frac{x}{2}=y-2$, and $\frac{x}{8}+\frac{y}{7}=x-13$.

Ans. $x=16$, $y=7$.

6. $\frac{y+x}{2}+\frac{y-x}{2}=x-\frac{2y}{3}$, and $x+y=16$.

Ans. $x=10$, $y=6$.

7. $\frac{2x-3y}{5}=x-2\frac{2}{5}$, $x-\frac{y-1}{2}=0$. *Ans.* $x=1$, $y=3$.

8. $2y+3x=y+43$, $y-\frac{x-4}{3}=y-\frac{x}{5}$.

Ans. $x=10$, $y=13$.

9. $4y-\frac{x-y}{2}=x+18$, and $27-y=x+y+4$.

Ans. $x=9$, $y=7$.

10. $1-\frac{y-x}{6}+4=y-16\frac{2}{3}$, $\frac{y}{5}-2=\frac{x}{5}$.

Ans. $x=10$, $y=20$.

116. Having explained the principal methods of elimination, we shall add a few examples which may be solved by any one of them; and often, indeed, it may be advantageous to employ them all, even in the same example.

General Exercises.

Find the values of x and y in the following simultaneous equations: —

1. $2x + 3y = 16$, and $3x - 2y = 11$. *Ans.* $x = 5$, $y = 2$.

2. $\frac{2x}{5} + \frac{3y}{4} = \frac{9}{20}$, and $\frac{3x}{4} + \frac{2y}{5} = \frac{61}{120}$.

Ans. $x = \frac{1}{2}$, $y = \frac{1}{3}$.

3. $\frac{x}{7} + 7y = 99$, and $\frac{y}{7} + 7x = 51$. *Ans.* $x = 7$, $y = 14$.

4. $\frac{x}{2} - 12 = \frac{y}{4} + 8$, $\frac{x+y}{5} + \frac{x}{3} - 8 = \frac{2y - x}{4} + 27$.

Ans. $x = 60$, $y = 40$.

5. $\left\{\begin{array}{l} x - \frac{1}{2}y + \frac{4x}{5} = 6\frac{4}{5}. \\ \frac{x-y}{2} + 7x = 41. \end{array}\right.$ *Ans.* $\left\{\begin{array}{l} x = 6. \\ y = 8. \end{array}\right.$

6. $\left\{\begin{array}{l} \frac{x-y}{4} + \frac{x+y}{5} = 2\frac{1}{10}. \\ \frac{1}{2}x - y + 4\frac{1}{4}y = 12\frac{1}{4}. \end{array}\right.$ *Ans.* $\left\{\begin{array}{l} x = 5. \\ y = 3. \end{array}\right.$

7. $\left\{\begin{array}{l} \frac{3y-x}{6} + \frac{2x-y}{4} = 5. \\ 6x - y + \frac{8-2x}{4} = 43\frac{1}{2}. \end{array}\right.$ *Ans.* $\left\{\begin{array}{l} x = 9. \\ y = 8. \end{array}\right.$

8. $\left\{\begin{array}{l} \frac{3x-8}{4} + \frac{y-6}{5} + y = 18\frac{7}{10}. \\ 8x - 3 - \frac{6-y}{3} = 79. \end{array}\right.$ *Ans.* $\left\{\begin{array}{l} x = 10. \\ y = 12. \end{array}\right.$

9. $\begin{cases} \dfrac{4x-4}{3} - \dfrac{y-5}{4} - 5 = 1\frac{2}{3}. \\ \dfrac{1}{2}x - \dfrac{1}{3}y + \dfrac{y-4}{3} = \dfrac{5}{3}. \end{cases}$ *Ans.* $\begin{cases} x = 6. \\ y = 5. \end{cases}$

10. $\begin{cases} ax - by = c. \\ a - y + x = d. \end{cases}$ *Ans.* $\begin{cases} x = \dfrac{c + ab - bd}{a - b}. \\ y = \dfrac{a^2 + c - ad}{a - b}. \end{cases}$

11. $\begin{cases} 13x + 7y - 341 = 7\frac{1}{2}y + 43\frac{1}{2}x. \\ 2x + \frac{1}{2}y = 1. \end{cases}$ *Ans.* $\begin{cases} x = -12. \\ y = 50. \end{cases}$

12. $\begin{cases} (x+5)(y-7) = (x+1)(y-9) - 112. \\ 2x + 10 = 3y + 1. \end{cases}$ *Ans.* $\begin{cases} x = 3. \\ y = 5. \end{cases}$

13. $\begin{cases} ax = by. \\ x + y = c. \end{cases}$ *Ans.* $\begin{cases} x = \dfrac{bc}{a+b}. \\ y = \dfrac{ac}{a+b}. \end{cases}$

14. $\begin{cases} ax + by = c. \\ fx + gy = h. \end{cases}$ *Ans.* $\begin{cases} x = \dfrac{cg - bh}{ag - bf}. \\ y = \dfrac{ah - cf}{ag - bf}. \end{cases}$

15. $\begin{cases} \dfrac{a}{b+y} = \dfrac{b}{3a+x}. \\ ax + 2by = d. \end{cases}$ *Ans.* $\begin{cases} x = \dfrac{2b^2 - 6a^2 + d}{3a}. \\ y = \dfrac{3a^2 - b^2 + d}{3b}. \end{cases}$

16. $\begin{cases} bcx = cy - 2b. \\ b^2y + \dfrac{a(c^3 - b^3)}{bc} = \dfrac{2b^3}{c} + c^2x. \end{cases}$ *Ans.* $\begin{cases} x = \dfrac{a}{bc}. \\ y = \dfrac{a + 2b}{c}. \end{cases}$

17. $\left\{\begin{aligned} 3x+5y &= \frac{(8b-2f)bf}{b^2-f^2}. \\ y-x &= \frac{-2bf^2}{b^2-f^2}. \end{aligned}\right\}$ *Ans.* $\left\{\begin{aligned} x &= \frac{bf}{b-f}. \\ y &= \frac{bf}{b+f}. \end{aligned}\right.$

PROBLEMS FOR SOLUTION.

1. What fraction is that to the numerator of which if 1 be added the value will be $\frac{1}{3}$, but if 1 be added to its denominator the value will be $\frac{1}{4}$?

Let	$\frac{x}{y}=$ the fraction.
Then	$\frac{x+1}{y}=\frac{1}{3}$, and $\frac{x}{y+1}=\frac{1}{4}$.
Whence	$3x+3=y$, and $4x=y+1$.
Subtracting,	$x-3=1$, and $x=4$.
Hence	$12+3=y$.
	$\therefore\ y=15$.

2. A market-woman bought a certain number of eggs at 2 for a penny, and as many others at 3 for a penny; and, having sold them altogether at the rate of 5 for 2 pence, she found that she had lost 4 pence. How many of both kinds did she buy?

Let	$2x=$ the whole number of eggs.
Then	$x=$ the number of eggs of each sort.
Then will	$\frac{1}{2}x=$ the cost of the first sort,
and	$\frac{1}{3}x=$ the cost of the second sort.

But by the conditions of the question

$$5:2x::2:\frac{4x}{5}.$$

Hence $\frac{4x}{5}$ will denote the amount for which the eggs were sold.

But by the conditions

$$\frac{1}{2}x + \frac{1}{3}x - \frac{4x}{5} = 4.$$

$$\therefore 15x + 10x - 24x = 120.$$

$$\therefore x = 120, \text{ the number of eggs of each sort.}$$

3. A person possessed a capital of \$30,000, for which he received a certain interest; but he owed the sum of \$20,000, for which he paid a certain annual interest. The interest that he received exceeded that which he paid by \$800. Another person possessed \$35,000, for which he received interest at the second of the above rates; but he owed \$24,000, for which he paid interest at the first of the above rates. The interest that he received annually exceeded that which he paid by \$310. Required the two rates of interest.

Let x denote the number of units in the first rate of interest, and y the unit in the second rate. Then each may be regarded as denoting the interest on \$100 for 1 year.

To obtain the interest of \$30,000 at the first rate, denoted by x, we form the proportion

$$100 : 30{,}000 :: x : \frac{30000x}{100}, \text{ or } 300x;$$

and for the interest of \$20,000, the rate being y,

$$100 : 20{,}000 :: y : \frac{20000y}{100}, \text{ or } 200y.$$

But by the conditions the difference between these two amounts is equal to \$800.

We have, then, for the first equation of the problem,

$$300x - 200y = 800.$$

By expressing algebraically the second condition of the problem, we obtain a second equation,

$$350y - 240x = 310.$$

Both members of the first equation being divisible by 100, and those of the second by 10, we have

$$3x - 2y = 8, \quad 35y - 24x = 31.$$

To eliminate x, multiply the first equation by 8, and then add the result to the second. There results

$$19y = 95, \text{ whence } y = 5.$$

Substituting for y, in the first equation, this value, and that equation becomes

$$3x - 10 = 8, \text{ whence } x = 6.$$

Therefore the first rate is 6 per cent; and the second, 5.

VERIFICATION.

$$\$30{,}000 \text{ at } 6 \text{ per cent} = 30{,}000 \times .06 = \$1800,$$

$$\$20{,}000 \text{ at } 5 \text{ per cent} = 20{,}000 \times .05 = \$1000;$$

and we have $1800 - 1000 = 800.$

The second condition can be verified in the same manner.

4. What two numbers are those whose difference is 7, and sum 33? *Ans.* 13 and 20.

5. Divide the number 75 into two such parts that three times the greater may exceed seven times the less by 15.

Ans. 54 and 21.

6. In a mixture of wine and cider, ½ of the whole, plus 25 gallons, was wine; and ⅓ part, minus 5 gallons, was cider. How many gallons were there of each?

Ans. 85 of wine, and 35 of cider.

7. A bill of £120 was paid in guineas and moidores, and the number of pieces used of both sorts was just 100. If the guinea be estimated at 21s., and the moidore at 27s., how many pieces were there of each sort? *Ans.* 50.

8. Two travelers set out at the same time from London and York, whose distance apart is 150 miles. One of them travels 8 miles a day; and the other, 7. In what time will they meet? *Ans.* In 10 days.

9. At a certain election 375 persons voted for two candidates, and the candidate chosen had a majority of 91. How many voted for each?

Ans. 233 for one, and 142 for the other.

10. A person has two horses, and a saddle worth £50. Now, if the saddle be put on the back of the first horse, it makes their joint value double that of the second horse; but if it be put on the back of the second, it makes their joint value triple that of the first. What is the value of each horse? *Ans.* One £30, and the other £40.

11. The hour and minute hands of a clock are exactly together at 12 o'clock. When will they again be together?

Ans. 1h. $5\frac{5}{11}$m.

12. A man and his wife usually drank a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days. How many days would the man alone be in drinking it? *Ans.* 20 days.

13. If 32 pounds of sea-water contain 1 pound of salt, how much fresh water must be added to these 32 pounds in order that the quantity of salt contained in 32 pounds of the new mixture shall be reduced to 2 ounces, or $\frac{1}{8}$ of a pound?

Ans. 224 lbs.

14. A person who possessed $100,000 placed the greater part of it out at 5 per cent interest, and the other at 4 per cent. The interest which he received for the whole amounted to $4640. Required the two parts. *Ans.* $64,000 and $36,000.

15. At the close of an election the successful candidate had a majority of 1500 votes. Had a fourth of the votes of the unsuccessful candidate been also given to him, he would have received three times as many as his competitor, wanting three thousand five hundred. How many votes did each receive? *Ans.* 1st, 6500; 2d, 5000.

16. A gentleman bought a gold and a silver watch, and a chain worth \$25. When he put the chain on the gold watch, it and the chain became worth three and a half times more than the silver watch; but when he put the chain on the silver watch, they became worth one half the gold watch and \$15 over. What was the value of each watch?

Ans. Gold watch, \$80; silver watch, \$30.

17. There is a certain number expressed by two figures, which figures are called digits. The sum of the digits is 11, and if 13 be added to the first digit the sum will be three times the second. What is the number? *Ans.* 56.

18. From a company of ladies and gentlemen, 15 ladies retire, and there are then left 2 gentlemen to each lady; after which 45 gentlemen depart, when there are left 5 ladies to each gentleman. How many were there of each at first?

Ans. 50 gentlemen, 40 ladies.

19. A person wishes to dispose of his horse by lottery. If he sells the tickets at \$2 each, he will lose \$30 on his horse; but if he sells them at \$3 each, he will receive \$30 more than his horse cost him. What is the value of the horse, and the number of tickets? *Ans.* Horse, \$150; No. of tickets, 60.

20. A person purchases a lot of wheat at \$1, and a lot of rye at 75 cents, per bushel, the whole costing him \$117.50. He then sells $\frac{1}{4}$ of his wheat and $\frac{1}{5}$ of his rye at the same rate, and realizes \$27.50. How much did he buy of each?

Ans. 80 bushels of wheat, 50 of rye.

21. There are 52 pieces of money in each of two bags. A takes from one, and B from the other. A takes twice as much as B left, and B takes seven times as much as A left. How much did each take? *Ans.* A, 48 pieces; B, 28 pieces.

22. Two persons, A and B, purchase a house together, worth \$1200. Says A to B, "Give me two thirds of your

money, and I can purchase it alone." But says B to A, "If you will give me three fourths of your money, I shall be able to purchase it alone." How much had each?

Ans. A, \$800; B, \$600.

23. A grocer finds that if he mixes sherry and brandy in the proportion of 2 to 1, the mixture will be worth 78s. per dozen; but if he mixes them in the proportion of 7 to 2, he can get 79s. a dozen. What is the price of each liquor per dozen? *Ans.* Sherry, 81s.; brandy, 72s.

Equations containing Three or More Unknown Quantities.

117. Let us now consider equations involving three or more unknown quantities.

Take the group of simultaneous equations,

$$5x - 6y + 4z = 15 \qquad (1)$$

$$7x + 4y - 3z = 19 \qquad (2)$$

$$2x + y + 6z = 46 \qquad (3)$$

To eliminate z by means of the first two equations, multiply the first by 3, and the second by 4. Then, since the coefficients of z have contrary signs, add the two results together. This gives a new equation,

$$43x - 2y = 121 \qquad (4)$$

Multiplying the second equation by 2 (a factor of the coefficient of z in the third equation), and adding the result to the third equation, we have

$$16x + 9y = 84 \qquad (5)$$

The question is then reduced to finding the values of x and y, which will satisfy the new equations (4) and (5).

Now, if the first be multiplied by 9, the second by 2, and the results added together, we find

$$419x = 1257, \text{ whence } x = 3.$$

We might, by means of Equations (4) and (5), determine y in the same way that we have determined x; but the value of y may be determined more simply, by substituting the value of x in Equation (5). Thus,

$$48 + 9y = 84. \qquad \therefore\ y = \frac{84 - 48}{9} = 4.$$

In the same manner the first of the three given equations becomes, by substituting the values of x and y,

$$15 - 24 + 4z = 15. \qquad \therefore\ z = \frac{24}{4} = 6.$$

In the same way any group of simultaneous equations may be solved. Hence the rule: —

Combine one equation of the group with each of the others, and eliminate one unknown quantity: there will result a new group containing one equation less than the original group.

Combine one equation of this new group with each of the others, and eliminate a second unknown quantity: there will result a new group containing two equations less than the original group.

Continue the operation until a single equation is found, containing but one unknown quantity.

Find the value of this unknown quantity by the preceding rules; substitute this in one of the group of two equations, and find the value of a second unknown quantity; substitute these in either of the groups of three, finding a third unknown quantity; and so on till the values of all are found.

NOTES. — 1. In order that the value of the unknown quantities may be determined, there must be just as many independent equations of condition as there are unknown quantities. If there are fewer equations than unknown quantities, the resulting equation will contain at least two unknown quantities, and hence their values cannot be found (§ 110). If there are more equations than unknown quantities, the conditions may be contradictory, and the equations impossible.

2. It often happens that each of the proposed equations does not contain all the unknown quantities. In this case the elimination may often be very readily performed.

Take the four equations involving four unknown quantities,

$$2x - 3y + 2z = 13 \qquad (1) \qquad\qquad 4y + 2z = 14 \qquad (3)$$

$$4u - 2x = 30 \qquad (2) \qquad\qquad 5y + 3u = 32 \qquad (4)$$

By inspecting these equations, we see that the elimination of z in the two equations (1) and (3) will give an equation involving x and y; and if we eliminate u in Equations (2) and (4), we shall obtain a second equation involving x and y. These last two unknown quantities may therefore be easily determined. In the first place, the elimination of z from (1) and (3) gives

$$7y - 2x = 1.$$

That of u from (2) and (4) gives

$$20y + 6x = 38.$$

Multiplying the first of these equations by 3, and adding,

$$41y = 41.$$

Whence $\qquad y = 1.$

Substituting this value in

$$7y - 2x = 1,$$

we find $\qquad x = 3.$

Substituting for x its value in Equation (2), it becomes

$$4u - 6 = 30.$$

Whence $\qquad u = 9.$

Substituting for y its value in Equation (3), there results

$$z = 5.$$

Exercises.

1. Given $\left\{\begin{aligned} x+\ y+\ z&=29,\\ x+2y+3z&=62,\\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z&=10,\end{aligned}\right\}$ to find x, y, and z.

Ans. $x=8,\ y=9,\ z=12.$

2. Given $\left\{\begin{aligned} 2x+4y-3z&=22,\\ 4x-2y+5z&=18,\\ 6x+7y-\ z&=63,\end{aligned}\right\}$ to find x, y, and z.

Ans. $x=3,\ y=7,\ z=4.$

3. Given $\left\{\begin{aligned} x+\frac{1}{2}y+\frac{1}{3}z&=32,\\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z&=15,\\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z&=12,\end{aligned}\right\}$ to find x, y, and z.

Ans. $x=12,\ y=20,\ z=30.$

4. Given $\left\{\begin{aligned} x+y+z&=29\tfrac{1}{4},\\ x+y-z&=18\tfrac{1}{4},\\ x-y+z&=13\tfrac{3}{4},\end{aligned}\right\}$ to find x, y, and z.

Ans. $x=16,\ y=7\tfrac{3}{4},\ z=5\tfrac{1}{2}.$

5. Given $\left\{\begin{aligned} 3x+5y&=161,\\ 7x+2z&=209,\\ 2y+\ z&=\ 89,\end{aligned}\right\}$ to find x, y, and z.

Ans. $x=17,\ y=22,\ z=45.$

6. Given $\left\{\begin{aligned} \frac{1}{x}+\frac{1}{y}&=a,\\ \frac{1}{x}+\frac{1}{z}&=b,\\ \frac{1}{y}+\frac{1}{z}&=c,\end{aligned}\right\}$ to find x, y, and z.

Ans. $x=\dfrac{2}{a+b-c},\ \ y=\dfrac{2}{a+c-b},\ \ z=\dfrac{2}{b+c-a}.$

Hence $\frac{4x}{5}$ will denote the amount for which the eggs were sold.

But by the conditions

$$\frac{1}{2}x + \frac{1}{3}x - \frac{4x}{5} = 4.$$

$$\therefore\ 15x + 10x - 24x = 120.$$

$$\therefore\ x = 120, \text{ the number of eggs of each sort.}$$

3. A person possessed a capital of \$30,000, for which he received a certain interest; but he owed the sum of \$20,000, for which he paid a certain annual interest. The interest that he received exceeded that which he paid by \$800. Another person possessed \$35,000, for which he received interest at the second of the above rates; but he owed \$24,000, for which he paid interest at the first of the above rates. The interest that he received annually exceeded that which he paid by \$310. Required the two rates of interest.

Let x denote the number of units in the first rate of interest, and y the unit in the second rate. Then each may be regarded as denoting the interest on \$100 for 1 year.

To obtain the interest of \$30,000 at the first rate, denoted by x, we form the proportion

$$100 : 30,000 : : x : \frac{30000x}{100}, \text{ or } 300x;$$

and for the interest of \$20,000, the rate being y,

$$100 : 20,000 : : y : \frac{20000y}{100}, \text{ or } 200y.$$

But by the conditions the difference between these two amounts is equal to \$800.

We have, then, for the first equation of the problem,

$$300x - 200y = 800.$$

By expressing algebraically the second condition of the problem, we obtain a second equation,

$$350y - 240x = 310.$$

Both members of the first equation being divisible by 100, and those of the second by 10, we have

$$3x - 2y = 8, \quad 35y - 24x = 31.$$

To eliminate x, multiply the first equation by 8, and then add the result to the second. There results

$$19y = 95, \text{ whence } y = 5.$$

Substituting for y, in the first equation, this value, and that equation becomes

$$3x - 10 = 8, \text{ whence } x = 6.$$

Therefore the first rate is 6 per cent; and the second, 5.

VERIFICATION.

$$\$30{,}000 \text{ at 6 per cent} = 30{,}000 \times .06 = \$1800,$$

$$\$20{,}000 \text{ at 5 per cent} = 20{,}000 \times .05 = \$1000;$$

and we have $$1800 - 1000 = 800.$$

The second condition can be verified in the same manner.

4. What two numbers are those whose difference is 7, and sum 33? *Ans.* 13 and 20.

5. Divide the number 75 into two such parts that three times the greater may exceed seven times the less by 15.

Ans. 54 and 21.

6. In a mixture of wine and cider, $\frac{1}{2}$ of the whole, plus 25 gallons, was wine; and $\frac{1}{3}$ part, minus 5 gallons, was cider. How many gallons were there of each?

Ans. 85 of wine, and 35 of cider.

7. A bill of £120 was paid in guineas and moidores, and the number of pieces used of both sorts was just 100. If the guinea be estimated at 21s., and the moidore at 27s., how many pieces were there of each sort? *Ans.* 50.

8. Two travelers set out at the same time from London and York, whose distance apart is 150 miles. One of them travels 8 miles a day; and the other, 7. In what time will they meet? *Ans.* In 10 days.

9. At a certain election 375 persons voted for two candidates, and the candidate chosen had a majority of 91. How many voted for each?

Ans. 233 for one, and 142 for the other.

10. A person has two horses, and a saddle worth £50. Now, if the saddle be put on the back of the first horse, it makes their joint value double that of the second horse; but if it be put on the back of the second, it makes their joint value triple that of the first. What is the value of each horse? *Ans.* One £30, and the other £40.

11. The hour and minute hands of a clock are exactly together at 12 o'clock. When will they again be together?

Ans. 1h. $5\frac{5}{11}$m.

12. A man and his wife usually drank a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days. How many days would the man alone be in drinking it? *Ans.* 20 days.

13. If 32 pounds of sea-water contain 1 pound of salt, how much fresh water must be added to these 32 pounds in order that the quantity of salt contained in 32 pounds of the new mixture shall be reduced to 2 ounces, or $\frac{1}{8}$ of a pound?

Ans. 224 lbs.

14. A person who possessed $100,000 placed the greater part of it out at 5 per cent interest, and the other at 4 per cent. The interest which he received for the whole amounted to $4640. Required the two parts. *Ans.* $64,000 and $36,000.

15. At the close of an election the successful candidate had a majority of 1500 votes. Had a fourth of the votes of the unsuccessful candidate been also given to him, he would have received three times as many as his competitor, wanting three thousand five hundred. How many votes did each receive? *Ans.* 1st, 6500; 2d, 5000.

16. A gentleman bought a gold and a silver watch, and a chain worth \$25. When he put the chain on the gold watch, it and the chain became worth three and a half times more than the silver watch; but when he put the chain on the silver watch, they became worth one half the gold watch and \$15 over. What was the value of each watch?

Ans. Gold watch, \$80; silver watch, \$30.

17. There is a certain number expressed by two figures, which figures are called digits. The sum of the digits is 11, and if 13 be added to the first digit the sum will be three times the second. What is the number? *Ans.* 56.

18. From a company of ladies and gentlemen, 15 ladies retire, and there are then left 2 gentlemen to each lady; after which 45 gentlemen depart, when there are left 5 ladies to each gentleman. How many were there of each at first?

Ans. 50 gentlemen, 40 ladies.

19. A person wishes to dispose of his horse by lottery. If he sells the tickets at \$2 each, he will lose \$30 on his horse; but if he sells them at \$3 each, he will receive \$30 more than his horse cost him. What is the value of the horse, and the number of tickets? *Ans.* Horse, \$150; No. of tickets, 60.

20. A person purchases a lot of wheat at \$1, and a lot of rye at 75 cents, per bushel, the whole costing him \$117.50. He then sells $\frac{1}{4}$ of his wheat and $\frac{1}{5}$ of his rye at the same rate, and realizes \$27.50. How much did he buy of each?

Ans. 80 bushels of wheat, 50 of rye.

21. There are 52 pieces of money in each of two bags. A takes from one, and B from the other. A takes twice as much as B left, and B takes seven times as much as A left. How much did each take? *Ans.* A, 48 pieces; B, 28 pieces.

22. Two persons, A and B, purchase a house together, worth \$1200. Says A to B, "Give me two thirds of your

money, and I can purchase it alone." But says B to A, "If you will give me three fourths of your money, I shall be able to purchase it alone." How much had each?

Ans. A, \$800; B, \$600.

23. A grocer finds that if he mixes sherry and brandy in the proportion of 2 to 1, the mixture will be worth 78s. per dozen; but if he mixes them in the proportion of 7 to 2, he can get 79s. a dozen. What is the price of each liquor per dozen? *Ans.* Sherry, 81s.; brandy, 72s.

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In the same way any group of simultaneous equations may be solved. Hence the rule: —

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By inspecting these equations, we see that the elimination of z in the two equations (1) and (3) will give an equation involving x and y; and if we eliminate u in Equations (2) and (4), we shall obtain a second equation involving x and y. These last two unknown quantities may therefore be easily determined. In the first place, the elimination of z from (1) and (3) gives

$$7y-2x=1.$$

That of u from (2) and (4) gives

$$20y+6x=38.$$

Multiplying the first of these equations by 3, and adding,

$$41y=41.$$

Whence $\qquad y=1.$

Substituting this value in

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we find $\qquad x=3.$

Substituting for x its value in Equation (2), it becomes

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Whence $\qquad u=9.$

Substituting for y its value in Equation (3), there results

$$z=5.$$

Exercises.

1. Given $\left\{\begin{array}{l} x+\ \ y+\ \ z=29, \\ x+2y+3z=62, \\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=10, \end{array}\right\}$ to find x, y, and z.

Ans. $x=8,\ y=9,\ z=12.$

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Ans. $x=3,\ y=7,\ z=4.$

3. Given $\left\{\begin{array}{l} x+\frac{1}{2}y+\frac{1}{3}z=32, \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15, \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12, \end{array}\right\}$ to find x, y, and z.

Ans. $x=12,\ y=20,\ z=30.$

4. Given $\left\{\begin{array}{l} x+y+z=29\frac{1}{4}, \\ x+y-z=18\frac{1}{4}, \\ x-y+z=13\frac{3}{4}, \end{array}\right\}$ to find x, y, and z.

Ans. $x=16,\ y=7\frac{3}{4},\ z=5\frac{1}{2}.$

5. Given $\left\{\begin{array}{l} 3x+5y=161, \\ 7x+2z=209, \\ 2y+\ \ z=\ \ 89, \end{array}\right\}$ to find x, y, and z.

Ans. $x=17,\ y=22,\ z=45.$

6. Given $\left\{\begin{array}{l} \frac{1}{x}+\frac{1}{y}=a, \\ \frac{1}{x}+\frac{1}{z}=b, \\ \frac{1}{y}+\frac{1}{z}=c, \end{array}\right\}$ to find x, y, and z.

Ans. $x=\dfrac{2}{a+b-c},\quad y=\dfrac{2}{a+c-b},\quad z=\dfrac{2}{b+c-a}.$

NOTE. — In this example we should not proceed to clear the equation of fractions, but subtract immediately the second equation from the first, and then add the third. We thus find the value of y.

PROBLEMS FOR SOLUTION.

1. Divide the number 90 into four such parts that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, shall be equal each to each.

NOTE. — This problem may be easily solved by introducing a new unknown quantity.

Let x, y, z, and u denote the required parts, and designate by m the several equal quantities which arise from the conditions. We shall then have

$$x + 2 = m, \quad y - 2 = m, \quad 2z = m, \quad \frac{u}{2} = m.$$

From which we find

$$x = m - 2, \quad y = m + 2, \quad z = \frac{m}{2}, \quad u = 2m.$$

By adding the equations,

$$x + y + z + u = m + m + \frac{m}{2} + 2m = 4\tfrac{1}{2}m.$$

Since, by the conditions of the problem, the first member is equal to 90, we have

$$4\tfrac{1}{2}m = 90, \text{ or } \frac{9}{2}m = 90.$$

Hence $\quad m = 20.$

Having the value of m, we easily find the other values; viz.,

$$x = 18, \quad y = 22, \quad z = 10, \quad u = 40.$$

2. There are three ingots, composed of different metals, mixed together. A pound of the first contains 7 ounces of silver, 3 ounces of copper, and 6 of pewter. A pound of the second contains 12 ounces of silver, 3 ounces of copper, and 1 of pewter. A pound of the third contains 4 ounces of silver, 7 ounces of copper, and 5 of pewter. It is required to find

how much it will take of each of the three ingots to form a fourth, which shall contain in a pound, 8 ounces of silver, $3\frac{3}{4}$ of copper, and $4\frac{1}{4}$ of pewter.

Let x, y, and z denote the number of ounces which it is necessary to take from the three ingots respectively, in order to form a pound of the required ingot. Since there are 7 ounces of silver in a pound (or 16 ounces) of the first ingot, it follows that one ounce of it contains $\frac{7}{16}$ of an ounce of silver, and consequently in a number of ounces denoted by x there is $\frac{7x}{16}$ ounces of silver. In the same manner we find that $\frac{12y}{16}$ and $\frac{4z}{16}$ denote the number of ounces of silver taken from the second and the third; but, from the enunciation, one pound of the fourth ingot contains 8 ounces of silver. We have, then, for the first equation,

$$\frac{7x}{16}+\frac{12y}{16}+\frac{4z}{16}=8;$$

or, clearing fractions, $7x+12y+4z=128.$

As respects the copper, we should find

$$3x+3y+7z=60;$$

and with reference to the pewter,

$$6x+y+5z=68.$$

As the coefficients of y in these three equations are the most simple, it is convenient to eliminate this unknown quantity first.

Multiplying the second equation by 4, and subtracting the first from it, member from member, we have

$$5x+24z=112.$$

Multiplying the third equation by 3, and subtracting the second from the resulting equation, we have

$$15x+8z=144.$$

Multiplying this last equation by 3, and subtracting the preceding one, we obtain

$$40x=320.$$

Whence $x=8.$

Substitute this value for x in the equation

$$15x+8z=144,$$

it becomes $120+8z=144.$

Whence $z=3.$

Lastly, the two values, $x = 8$, $z = 3$, being substituted in the equation

$$6x + y + 5z = 68,$$

give
$$48 + y + 15 = 68.$$

Whence
$$y = 5.$$

Therefore, in order to form a pound of the fourth ingot, we must take 8 ounces of the first, 5 ounces of the second, and 3 of the third.

Verification. If there be 7 ounces of silver in 16 ounces of the first ingot, in 8 ounces of it there should be a number of ounces of silver expressed by

$$\frac{7 \times 8}{16}.$$

In like manner,
$$\frac{12 \times 5}{16} \text{ and } \frac{4 \times 3}{16}$$

will express the quantity of silver contained in 5 ounces of the second ingot, and 3 ounces of the third.

Now, we have

$$\frac{7 \times 8}{16} + \frac{12 \times 5}{16} + \frac{4 \times 3}{16} = \frac{128}{16} = 8.$$

Therefore a pound of the fourth ingot contains 8 ounces of silver, as required by the enunciation. The same conditions may be verified with respect to the copper and the pewter.

3. A's age is double B's, and B's is triple C's, and the sum of all their ages is 140. What is the age of each?

Ans. A's, 84; B's, 42; and C's, 14.

4. A person bought a chaise, horse, and harness for £60. The horse came to twice the price of the harness; and the chaise, to twice the cost of the horse and harness. What did he give for each?

Ans. £13 6s. 8d. for the horse; £6 13s. 4d. for the harness; £40 for the chaise.

5. Divide the number 36 into three such parts that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, may be all equal to each other.

Ans. 8, 12, and 16.

6. If A and B together can do a piece of work in 8 days, A and C together in 9 days, and B and C in 10 days, how many days would it take each to perform the same work alone? *Ans.* A, $14\frac{34}{49}$ days; B, $17\frac{23}{41}$; C, $23\frac{7}{31}$.

7. Three persons, A, B, and C, begin to play together, having among them all $600. At the end of the first game A has won one half of B's money, which, added to his own, makes double the amount B had at first. In the second game A loses and B wins just as much as C had at the beginning, when A leaves off with exactly what he had at first. How much had each at the beginning?

Ans. A, $300; B, $200; C, $100.

8. Three persons, A, B, and C, together possess $3640. If B gives A $400 of his money, then A will have $320 more than B; but if B takes $140 of C's money, then B and C will have equal sums. How much has each?

Ans. A, $800; B, $1280; C, $1560.

9. Three persons have a bill to pay, which neither alone is able to discharge. A says to B, "Give me the fourth of your money, and then I can pay the bill." B says to C, "Give me the eighth of yours, and I can pay it." But C says to A, "You must give me the half of yours before I can pay it, as I have but $8." What was the amount of their bill, and how much money had A and B?

Ans. Amount of the bill, $13; A had $10, and B $12.

10. A person possessed a certain capital, which he placed out at a certain interest. Another person, who possessed $10,000 more than the first, and who put out his capital 1 per cent more advantageously, had an annual income greater by $800. A third person, who possessed $5000 more than the first, putting out his capital 2 per cent more advantageously,

had an annual income greater by \$1500. Required the capitals of the three persons, and the rates of interest.

Ans. Capitals, \$30,000, \$40,000, \$45,000; rates of interest, 4%, 5%, 6%.

11. A widow receives an estate of \$15,000 from her deceased husband, with directions to divide it among two sons and three daughters so that each son may receive twice as much as each daughter, she herself to receive \$1000 more than all the children together. What was her share, and what the share of each child?

Ans. The widow's share, \$8000; each son's, \$2000; each daughter's, \$1000.

12. A certain sum of money is to be divided between three persons, A, B, and C. A is to receive \$3000 less than half of it; B, \$1000 less than one third part; and C, \$800 more than the fourth part of the whole. What is the sum to be divided, and what does each receive?

Ans. Sum, \$38,400; A receives \$16,200; B, \$11,800; C, \$10,400.

13. A person has three horses, and a saddle which is worth \$220. If the saddle be put on the back of the first horse, it will make his value equal to that of the second and third; if it be put on the back of the second, it will make his value double that of the first and third; if it be put on the back of the third, it will make his value triple that of the first and second. What is the value of each horse?

Ans. 1st, \$20; 2d, \$100; 3d, \$140.

14. The crew of a ship consisted of her complement of sailors, and a number of soldiers. There were 22 sailors to every three guns, and 10 over; also the whole number of hands was five times the number of soldiers and guns together. But after an engagement, in which the slain were

one fourth of the survivors, there wanted 5 men to make 13 men to every two guns. Required the number of guns, soldiers, and sailors.

Ans. 90 guns, 55 soldiers, and 670 sailors.

15. Three persons have $96, which they wish to divide equally between them. In order to do this, A, who has the most, gives to B and C as much as they have already; then B divides with A and C in the same manner, that is, by giving to each as much as he had after A had divided with them; C then makes a division with A and B; when it is found that they all have equal sums. How much had each at first?

Ans. 1st, $52; 2d, $28; 3d, $16.

16. Divide the number a into three such parts that the first shall be to the second as m to n, and the second to the third as p to q.

Ans. $x=\dfrac{amp}{mp+np+nq}$, $y=\dfrac{anp}{mp+np+nq}$, $z=\dfrac{anq}{mp+np+nq}$.

17. Three masons, A, B, and C, are to build a wall. A and B together can do it in 12 days; B and C, in 20 days; and A and C, in 15 days. In what time can each do it alone, and in what time can they all do it if they work together?

Ans. A, in 20 days; B, in 30; and C, in 60; all, in 10.

INEQUALITIES.

117a. An **inequality** is an algebraic expression of two unequal quantities, connected by the sign of inequality. Thus, $a > b$ is an inequality, showing that a is greater than b.

Of two negative quantities, that one is the greater *algebraically* which has the fewer units.

The part on the left of the sign is called the **first member**, and the part on the right the **second member**, of the inequality.

Two inequalities subsist in the *same* sense when the greater

quantity is in the first member of both or in the second member of both; they subsist in a *contrary* sense when the greater quantity is in the first member of one and in the second member of the other. Thus, the inequalities

$$35 > 30 \text{ and } 18 > 10$$

subsist in the same sense, and the inequalities

$$15 > 13 \text{ and } 12 < 14$$

subsist in a contrary sense.

The following principles enable us to transform inequalities: —

(1) *If we add the same quantity to, or subtract the same quantity from, both members of an inequality, the resulting inequality will subsist in the same sense.*

Thus, if we add 5 to, and subtract 5 from, both members of the inequality

$$4 > 2,$$

we have $$9 > 7 \text{ and } -1 > -3.$$

This principle enables us to transpose a term from one member of an inequality to the other by simply changing its sign. Thus, from the inequality

$$3x - b > 2x + a,$$

we find, by transposition,

$$x > a + b.$$

(2) *If two members of an inequality be multiplied or divided by a positive quantity, the resulting inequality will subsist in the same sense.*

Thus, if we multiply or divide both members of the inequality

$$12 > 8$$

by $+4$, we have $$48 > 32 \text{ and } 3 > 2.$$

This principle enables us to clear an inequality of fractions. Thus, if we multiply both members of the inequality

$$\frac{3x}{4}-\frac{2b}{3}>\frac{x}{6}-7$$

by 12, and reduce, we have

$$9x-8b>2x-84.$$

The following precautions are to be observed in treating inequalities: —

(3) *If both members of an inequality are positive, they may be raised to like powers without changing the sense of the inequality.*

Thus, if both members of the inequality

$$12>5$$

be squared or cubed, we have

$$144>25 \text{ and } 1728>125.$$

(4) *If the two members of an inequality be multiplied or divided by a negative quantity, the sense of the inequality must be reversed.*

Thus, if we multiply and divide both members of the inequality

$$3<6$$

by -3, we have

$$-9>-18 \text{ and } -1>-2.$$

(5) *If both members of an inequality are negative, they may be raised to any power of an odd degree, and the resulting inequality will subsist in the same sense; but, if both members be raised to a power of an even degree, the sense of the resulting inequality will be reversed.*

Thus, if we square and cube both members of the inequality

$$-3>-5,$$

we have $$9<25 \text{ and } -27>-125.$$

By the aid of these principles we may solve an inequality; that is, we can find an inequality in which the unknown quantity shall form one member, and a known quantity the other.

Exercises.

1. $5x-6>19$. *Ans.* $x>5$.

2. $3x+\frac{14}{2}x-30>10$. *Ans.* $x>4$.

3. $\frac{x}{6}-\frac{1}{3}x+\frac{x}{2}+\frac{13}{2}>\frac{17}{2}$. *Ans.* $x>6$.

4. $\frac{ax}{5}+bx-ab>\frac{a^2}{5}$. *Ans.* $x>a$.

5. $\frac{bx}{7}-ax+ab<\frac{b^2}{7}$. *Ans.* $x<b$.

CHAPTER VI.

POWERS.

118. A **power** of a quantity is the product obtained by taking that quantity any number of times as a factor.

If the quantity be taken once as a factor, we have the first power; if taken twice, we have the second power; if three times, the third power; if n times, the nth power, n being any whole number whatever.

A power is indicated by means of the exponential sign. Thus,

$a = a^1$	denotes	first power of a.
$a \times a = a^2$	"	square, or second power, of a.
$a \times a \times a = a^3$	"	cube, or third power, of a.
$a \times a \times a \times a = a^4$	"	fourth power of a.
$a \times a \times a \times a \times a = a^5$	"	fifth power of a.
$a \times a \times a \times a \ldots\ldots = a^m$	"	mth power of a.

NOTE. — Since $a^0 = 1$ (§ 49), $a^0 \times a = 1 \times a = a^1$; so that the two factors of a^1 are 1 and a.

In every power there are three things to be considered: —

(1) The quantity which enters as a factor, and which is called the **first power.**

(2) The small figure which is placed at the right of and a little above the letter, which is called the **exponent** of the power, and shows how many times the letter enters as a factor.

(3) The power itself, which is the final product, or result of the multiplications.

POWERS OF MONOMIALS.

119. Let it be required to raise the monomial $2a^3b^2$ to the fourth power. We have

$$(2a^3b^2)^4 = 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2,$$

which merely expresses that the fourth power is equal to the product which arises from taking the quantity four times as a factor. By the rules for multiplication, this product is

$$(2a^3b^2)^4 = 2^4a^{3+3+3+3}b^{2+2+2+2} = 2^4a^{12}b^8,$$

from which we see

(1) That the coefficient 2 must be raised to the fourth power; and

(2) That the exponent of each letter must be multiplied by 4, the exponent of the power.

As the same reasoning applies to every example, we have, for the raising of monomials to any power, the following rule:—

Raise the coefficient to the required power.

Multiply the exponent of each letter by the exponent of the power.

(1) What is the square of $3a^2y^3$? *Ans.* $9a^4y^6$.

(2) What is the cube of $6a^5y^2x$? *Ans.* $216a^{15}y^6x^3$.

(3) What is the fourth power of $2a^3y^3b^5$? *Ans.* $16a^{12}y^{12}b^{20}$.

(4) What is the square of $a^2b^5y^3$? *Ans.* $a^4b^{10}y^6$.

(5) What is the seventh power of a^2bcd^3? *Ans.* $a^{14}b^7c^7d^{21}$.

(6) What is the sixth power of $a^2b^3c^2d$? *Ans.* $a^{12}b^{18}c^{12}d^6$.

(7) What are the square and the cube of $-2a^2b^2$?

Square.	*Cube.*
$-2a^2b^2$	$-2a^2b^2$
$-2a^2b^2$	$-2a^2b^2$
$+4a^4b^4$	$+4a^4b^4$
	$-2a^2b^2$
	$-8a^6b^6$

By observing the way in which the powers are formed, we may conclude that

When the monomial is positive, all the powers will be positive.

When the monomial is negative, all even powers will be positive, and all odd powers will be negative.

Exercises.

1. What is the square of $-2a^4b^5$? *Ans.* $4a^8b^{10}$.
2. What is the cube of $-5a^nb^2$? *Ans.* $-125a^{3n}b^6$.
3. What is the eighth power of $-a^3xy^2$? *Ans.* $+a^{24}x^8y^{16}$.
4. What is the seventh power of $-a^mb^nc$? *Ans.* $-a^{7m}b^{7n}c^7$.
5. What is the sixth power of $2ab^6y^5$? *Ans.* $64a^6b^{36}y^{30}$.
6. What is the ninth power of $-a^nbc^2$? *Ans.* $-a^{9n}b^9c^{18}$.
7. What is the sixth power of $-3ab^2d$? *Ans.* $729a^6b^{12}d^6$.
8. What is the square of $-10a^mb^nc^3$? *Ans.* $100a^{2m}b^{2n}c^6$.
9. What is the cube of $-9a^mb^nd^3f^2$? *Ans.* $-729a^{3m}b^{3n}d^9f^6$.

10. What is the fourth power of $-4a^5b^3c^4d^5$?

Ans. $256a^{20}b^{12}c^{16}d^{20}$.

11. What is the cube of $-4a^{2m}b^{2n}c^3d$? *Ans.* $-64a^{6m}b^{6n}c^9d^3$.

12. What is the fifth power of $2a^3b^2xy$? *Ans.* $32a^{15}b^{10}x^5y^5$.

13. What is the square of $20x^ny^mc^5$? *Ans.* $400x^{2n}y^{2m}c^{10}$.

14. What is the fourth power of $3a^nb^{2n}c^3$? *Ans.* $81a^{4n}b^{8n}c^{12}$.

15. What is the fifth power of $-c^nd^{3m}x^2y^2$?

Ans. $-c^{5n}d^{15m}x^{10}y^{10}$.

16. What is the sixth power of $-a^nb^{2n}c^m$? *Ans.* $a^{6n}b^{12n}c^{6m}$.

17. What is the fourth power of $-2a^2c^2d^3$? *Ans.* $16a^8c^8d^{12}$.

Powers of Fractions.

120. From the definition of a power, and the rule for the multiplication of fractions, the cube of the fraction $\frac{a}{b}$ is written

$$\left(\frac{a}{b}\right)^3=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a^3}{b^3};$$

and since any fraction raised to any power may be written under the same form, we find any power of a fraction by the following rule: —

Raise the numerator to the required power, for a new numerator; and the denominator to the required power, for a new denominator.

The rule for signs is the same as in the last section.

Exercises.

Find the powers of the following fractions:—

1. $\left(\frac{a-c}{b+c}\right)^2.$ *Ans.* $\frac{a^2-2ac+c^2}{b^2+2bc+c^2}.$

2. $\left(\frac{xy}{3bc}\right)^3.$ *Ans.* $\frac{x^3y^3}{27b^3c^3}.$

3. $\left(\frac{-x^2y}{2ab}\right)^4.$ *Ans.* $\frac{x^8y^4}{16a^4b^4}.$

4. $\left(\frac{2ax^2y}{3bc^2}\right)^2.$ *Ans.* $\frac{4a^2x^4y^2}{9b^2c^4}.$

5. $\left(-\frac{dx}{3y^2}\right)^3.$ *Ans.* $-\frac{d^3x^3}{27y^6}.$

6. $\left(\frac{axy^3}{2bz^2}\right)^3.$ *Ans.* $\frac{a^3x^3y^9}{8b^3z^6}.$

7. $\left(-\frac{3ay^4}{2b^2x}\right)^4.$ *Ans.* $\frac{81a^4y^{16}}{16b^8x^4}.$

8. Fourth power of $\frac{ab^2c}{2x^2y^2}.$ *Ans.* $\frac{a^4b^8c^4}{16x^8y^8}.$

9. Cube of $\frac{x-y}{x+y}.$ *Ans.* $\frac{x^3-3x^2y+3xy^2-y^3}{x^3+3x^2y+3xy^2+y^3}.$

10. Fourth power of $\frac{2a^mx^n}{4a^py^q}.$ *Ans.* $\frac{a^{4m}x^{4n}}{16a^{4p}y^{4q}}.$

11. Fifth power of $-\frac{9bc^nx^m}{18y^pz^q}.$ *Ans.* $-\frac{b^5c^{5n}x^{5m}}{32y^{5p}z^{5q}}.$

POWERS OF BINOMIALS.

121. A **binomial,** like a monomial, may be raised to any power by the process of continued multiplication.

(1) Find the fifth power of the binomial $a+b$.

$$
\begin{array}{llllllll}
a & +b & & & & & & \text{1st power.} \\
a & +b & & & & & & \\
\hline
a^2 & +ab & & & & & & \\
 & +ab & +b^2 & & & & & \\
\hline
a^2 & +2ab & +b^2 & & & & & \text{2d power.} \\
a & +b & & & & & & \\
\hline
a^3 & +2a^2b & +ab^2 & & & & & \\
 & +a^2b & +2ab^2 & +b^3 & & & & \\
\hline
a^3 & +3a^2b & +3ab^2 & +b^3 & & & & \text{3d power.} \\
a & +b & & & & & & \\
\hline
a^4 & +3a^3b & +3a^2b^2 & +ab^3 & & & & \\
 & +a^3b & +3a^2b^2 & +3ab^3 & +b^4 & & & \\
\hline
a^4 & +4a^3b & +6a^2b^2 & +4ab^3 & +b^4 & & & \text{4th power.} \\
a & +b & & & & & & \\
\hline
a^5 & +4a^4b & +6a^3b^2 & +4a^2b^3 & +ab^4 & & & \\
 & +a^4b & +4a^3b^2 & +6a^2b^3 & +4ab^4 & +b^5 & & \\
\hline
a^5 & +5a^4b & +10a^3b^2 & +10a^2b^3 & +5ab^4 & +b^5 & & \textit{Ans.}
\end{array}
$$

122. NOTE.—It will be observed that the number of multiplications is always 1 less than the number of units in the exponent of the power. Thus, if the exponent is 1, no multiplication is necessary. If it is 2, we multiply once; if it is 3, twice; if 4, three times; etc. The powers of polynomials may be expressed by means of an exponent. Thus, to express that $a+b$ is to be raised to the fifth power, we write $(a+b)^5$; if to the mth power, we write $(a+b)^m$.

(2) Find the fifth power of the binomial $a - b$.

$$
\begin{array}{llllll l}
a & -b & & & & & \text{1st power.} \\
a & -b & & & & & \\
\hline
a^2 & -\ ab & & & & & \\
 & -\ ab & +b^2 & & & & \\
\hline
a^2 & -2ab & +b^2 & & & & \text{2d power.} \\
a & -b & & & & & \\
\hline
a^3 & -2a^2b & +\ ab^2 & & & & \\
 & -\ a^2b & +2ab^2 & -b^3 & & & \\
\hline
a^3 & -3a^2b & +3ab^2 & -b^3 & & & \text{3d power.} \\
a & -b & & & & & \\
\hline
a^4 & -3a^3b & +3a^2b^2 & -\ ab^3 & & & \\
 & -\ a^3b & +3a^2b^2 & -3ab^3 & +b^4 & & \\
\hline
a^4 & -4a^3b & +6a^2b^2 & -4ab^3 & +b^4 & & \text{4th power.} \\
a & -b & & & & & \\
\hline
a^5 & -4a^4b & +6a^3b^2 & -4a^2b^3 & +\ ab^4 & & \\
 & -\ a^4b & +4a^3b^2 & -6a^2b^3 & +4ab^4 & -b^5 & \\
\hline
a^5 & -5a^4b & +10a^3b^2 & -10a^2b^3 & +5ab^4 & -b^5 & \textit{Ans.}
\end{array}
$$

In the same way the higher powers may be obtained. By examining the powers of these binomials, it is plain that four things must be considered, —

I. The number of terms of the power.
II. The signs of the terms.
III. The exponents of the letters.
IV. The coefficients of the terms.

Let us see according to what laws these are formed.

I. The Number of Terms.

123. By examining the several multiplications, we shall observe that the first power of a binomial contains two terms; the second power, three terms; the third power, four terms;

the fourth power, five; the fifth power, six; etc. Hence we may conclude that

The number of terms in any power of a binomial is greater by one than the exponent of the power.

II. The Signs of the Terms.

124. It is evident that when both terms of the given binomial are plus, *all the terms of the power will be plus.*

If the second term of the binomial is negative, then *all the odd terms, counted from the left, will be positive, and all the even terms negative.*

III. The Exponents.

125. The letter which occupies the first place in a binomial is called the **leading letter.** Thus, a is the leading letter in the binomials $a+b$ and $a-b$.

(1) It is evident that the exponent of the leading letter in the first term will be the same as the exponent of the power, and that this exponent will diminish by one in each term to the right until we reach the last term, when it will be 0 (§ 49).

(2) The exponent of the second letter is 0 in the first term, and increases by one in each term to the right, to the last term, when the exponent is the same as that of the given power.

(3) The sum of the exponents of the two letters in any term is equal to the exponent of the given power. This last remark will enable us to verify any result obtained by means of the binomial formula.

Let us now apply these principles in the two following examples, in which the coefficients are omitted: —

$$(a+b)^6 \quad \ldots \quad a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6,$$

$$(a-b)^6 \quad \ldots \quad a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6.$$

As the pupil should be practiced in writing the terms with their proper signs without the coefficients, we will add a few more examples in which the coefficients are not given.

1. $(a+b)^3$ . . . $a^3+a^2b+ab^2+b^3$.
2. $(a-b)^4$ . . . $a^4-a^3b+a^2b^2-ab^3+b^4$.
3. $(a+b)^5$ . . . $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$.
4. $(a-b)^7$ . . . $a^7-a^6b+a^5b^2-a^4b^3+a^3b^4-a^2b^5+ab^6-b^7$.

IV. The Coefficients.

126. The coefficient of the first term is 1. The coefficient of the second term is the same as the exponent of the given power. The coefficient of the third term is found by multiplying the coefficient of the second term by the exponent of the leading letter in that term, and dividing the product by 2. Finally,

If the coefficient of any term be multiplied by the exponent of the leading letter in that term, and the product divided by the number which marks the place of the term from the left, the quotient will be the coefficient of the next term.

Thus, to find the coefficients in the example

$$(a-b)^7 \;.\;.\;.\; a^7-a^6b+a^5b^2-a^4b^3+a^3b^4-a^2b^5+ab^6-b^7,$$

we first place the exponent 7 as a coefficient of the second term. Then, to find the coefficient of the third term, we multiply 7 by 6, the exponent of a, and divide by 2. The quotient, 21, is the coefficient of the third term. To find the coefficient of the fourth, we multiply 21 by 5, and divide the product by 3: this gives 35. To find the coefficient of the fifth term, we multiply 35 by 4, and divide the product by 4: this gives 35. The coefficient of the sixth term, found in the same way, is 21; that of the seventh, 7; and that of the eighth, 1. Collecting these coefficients,

$$(a-b)^7=a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7.$$

NOTE. — We see, in examining this last result, that the *coefficients of the extreme terms are each* 1, *and that the coefficients of terms equally distant from the extreme terms are equal.* It will therefore be sufficient to find the coefficients of the first half of the terms, and from these the others may be immediately written.

Exercises.

1. Find the fourth power of $a + b$.

Ans. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

2. Find the fourth power of $a - b$.

Ans. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.

3. Find the fifth power of $a + b$.

Ans. $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

4. Find the fifth power of $a - b$.

Ans. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.

5. Find the sixth power of $a + b$.

Ans. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.

6. Find the sixth power of $a - b$.

Ans. $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.

127. When the terms of the binomial have coefficients, we may still write out any power of it by means of the binomial formula.

(1) Let it be required to find the cube of $2c + 3d$.

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Here $2c$ takes the place of a in the formula, and $3d$ the place of b. Hence we have

$$(2c + 3d)^3 = (2c)^3 + 3.(2c)^2.3d + 3(2c)(3d)^2 + (3d)^3 \qquad (1)$$

and by performing the indicated operations we have

$$(2c + 3d)^3 = 8c^3 + 36c^2d + 54cd^2 + 27d^3 \qquad (2)$$

If we examine the second member of Equation (1), we see that each term is made up of three factors, — first, the numerical factor; second, some power of $2c$; and, third, some power of $3d$. The powers of $2c$ are arranged in descending order towards the right, the last term involving the 0 power of $2c$, or 1; the powers of $3d$ are arranged in ascending order from the first term, where the 0 power enters, to the last term.

The operation of raising a binomial involving coefficients is most readily effected by writing the three factors of each term in a vertical column, and then performing the multiplications as indicated below.

Find by this method the cube of $2c+3d$.

$$
\begin{array}{rlllll}
 & 1 + & 3 & + 3 & + 1 & \text{Coefficients.} \\
 & 8c^3 + & 4c^2 & + 2c & + 1 & \text{Powers of } 2c. \\
 & 1 + & 3d & + 9d^2 & + 27d^3 & \text{Powers of } 3d. \\
\hline
(2c+d)^3 = & 8c^3 + & 36c^2d & + 54cd^2 & + 27d^3 &
\end{array}
$$

The preceding operation hardly requires explanation. In the first line write the numerical coefficients corresponding to the particular power, in the second line write the descending powers of the leading term to the 0 power, and in the third line write the ascending powers of the following term from the 0 power upwards. It will be easiest to commence the second line on the right hand. The multiplication should be performed from above downwards.

(2) Find the fourth power of $3a^2c-2bd$.

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

$$
\begin{array}{lllll}
1 & + 4 & + 6 & + 4 & + 1 \\
81a^8c^4 & + 27a^6c^3 & + 9a^4c^2 & + 3a^2c & + 1 \\
1 & - 2bd & + 4b^2d^2 & - 8b^3d^3 & + 16b^4d \\
\hline
81a^8c^4 & - 216a^6c^3bd & + 216a^4c^2b^2d^2 & - 96a^2cb^3d^3 & + 16b^4d^4
\end{array}
$$

Exercises.

1. What is the cube of $3x - 6y$?

   Ans. $27x^3 - 162x^2y + 324xy^2 - 216y^3$.

2. What is the fourth power of $a - 3b$?

   Ans. $a^4 - 12a^3b + 54a^2b^2 - 108ab^3 + 81b^4$.

3. What is the fifth power of $c - 2d$?

   Ans. $c^5 - 10c^4d + 40c^3d^2 - 80c^2d^3 + 80cd^4 - 32d^5$.

4. What is the cube of $5a - 3d$?

   Ans. $125a^3 - 225a^2d + 135ad^2 - 27d^3$.

CHAPTER VII.

EXTRACTION OF ROOTS, AND RADICALS.

EXTRACTION OF ROOTS.

128. Evolution is the process of finding the equal factors of a number or quantity. One of such factors is called a **root** of the quantity, and the operation of finding it is called **extracting** the root.

The sign $\sqrt{\ }$ is called the **radical sign.** A number placed above and in the opening of the radical sign indicates the root to be extracted, and is called the **index of the root.** Thus, in $\sqrt[3]{81}$, 3 is the index of the root, and shows that the cube root is to be found; in $\sqrt[5]{32a^{10}b^5}$, 5 is the index of the root. When the index is 2, it may be omitted.

Instead of the radical sign and index to denote a root, a fractional exponent is often used, in which the index is the denominator. Thus, instead of $\sqrt[3]{a}$, we may use $a^{\frac{1}{3}}$ to denote the cube root of a; also $\sqrt[5]{a^2} = a^{\frac{2}{5}}$, $\sqrt[4]{b^3} = b^{\frac{3}{4}}$, etc.

The **square root** of a number is one of its two equal factors. Thus, $6 \times 6 = 36$: therefore 6 is the square root of 36.

The symbol for the square root is $\sqrt{\ }$ or the fractional exponent $\frac{1}{2}$. Thus,

$$\sqrt{a}, \text{ or } a^{\frac{1}{2}},$$

indicates the square root of a, or that one of the two equal factors of a is to be found.

Square Root.

129. A perfect square is any number which can be resolved into *two equal integral factors.*

The following table, verified by actual multiplication, indicates all the perfect squares between 1 and 100 : —

Squares . . .	1,	4,	9,	16,	25,	36,	49,	64,	81,	100.
Roots	1,	2,	3,	4,	5,	6,	7,	8,	9,	10.

We may employ this table for finding the square root of any perfect square between 1 and 100.

Look for the number in the first line. If it is found there, its square root will be found immediately under it.

If the given number is less than 100, and not a perfect square, *it will fall between two numbers of the upper line, and its square root will be found between the two numbers directly below. The lesser of the two will be the entire part of the root, and will be the true root to within less than* 1.

Thus, if the given number is 55, it is found between the perfect squares 49 and 64, and its root is 7 and a decimal fraction.

NOTE. — There are ten perfect squares between 1 and 100, if we include both numbers; and eight, if we exclude both.

If a number is greater than 100, its square root will be greater than 10; that is, it will contain *tens* and *units*. Let N denote such a number, x the tens of its square root, and y the units. Then will

$$N = (x+y)^2 = x^2 + 2xy + y^2 = x^2 + (2x+y)y;$$

that is, the number is equal to the *square of the tens* in its roots, plus *twice the product of the tens by the units*, plus *the square of the units.*

For example, extract the square root of 6084.

Since this number is composed of more than two places of figures, its root will contain more than one: but since it is less than 10,000, which is the square of 100, the root will contain but two figures; that is, units and tens.

$$6\dot{0}\ 8\dot{4}$$

Now, the square of the tens must be found in the two left-hand figures, which we will separate from the other two by putting a point over the place of units, and a second over the place of hundreds. These parts, of two figures each, are called **periods.** The part 60 is comprised between the two squares 49 and 64, of which the roots are 7 and 8: hence 7 *expresses the number of tens sought;* and the required root is composed of 7 tens and a certain number of units.

The figure 7 being found, we write it on the right of the given number, from which we separate it by a vertical line: then we subtract its square, 49, from 60, which leaves a remainder of 11, to which we bring down the two next figures, 84. The result of this operation, 1184, contains *twice the product of the tens by the units, plus the square of the units.*

$$\begin{array}{r|l|l}
 & 6\dot{0}\ 8\dot{4} & 78 \\
 & 49 & \\ \hline
7 \times 2 = 148 & 118\,4 & \\
 & 118\,4 & \\ \hline
 & \quad\ \ 0 &
\end{array}$$

But since tens multiplied by units cannot give a product of a less unit than tens, it follows that the last figure, 4, can form no part of the double product of the tens by the units. This double product is therefore found in the part 118, which we separate from the units' place, 4.

Now, if we double the tens, which gives 14, and then divide 118 by 14, the quotient 8 *will express the units*, or a number greater than the units. This quotient can never be too small, since the part 118 will be at least equal to twice the product of the tens by the units; but it may be too large, for the 118, besides the double product of the tens by the units, may likewise contain tens arising from the square of the units. To ascertain if the quotient 8 expresses the right number of units, we write the 8 on the right of the 14, which gives 148, and then we multiply 148 by 8. This multiplication, being effected, gives for a product 1184, a number equal

to the result of the first operation. Having subtracted the product, we find the remainder equal to 0: hence 78 is the root required. In this operation we form, first, the square of the tens; second, the double product of the tens by the units; and, third, the square of the units.

Indeed, in the operations, we have merely subtracted from the given number 6084: first, the square of 7 tens, or of 70; second, twice the product of 70 by 8; and, third, the square of 8, that is, the three parts which enter into the composition of the square, $70+8$, or 78; and since the result of the subtraction is 0, it follows that 78 is the square root of 6084.

130. The operations in the last example have been performed on but two periods; but it is plain that the same methods of reasoning are equally applicable to larger numbers, for, by changing the order of the units, we do not change the relation in which they stand to each other.

Thus, in the number 60 84 95, the two periods 60 84 have the same relation to each other as in the number 60 84; and hence the methods used in the last example are equally applicable to larger numbers.

131. Hence, for the extraction of the square root of numbers, we have the following rule: —

Point off the given number into periods of two figures each, beginning at the right hand.

Note the greatest perfect square in the first period on the left, and place its root on the right, after the manner of a quotient in division; then subtract the square of this root from the first period, and bring down the second period for a remainder.

Double the root already found, and place the result on the left for a divisor. Seek how many times the divisor is contained in the remainder, exclusive of the right-hand figure, and place the figure in the root and also at the right of the divisor.

Multiply the divisor thus augmented by the last figure of the root, and subtract the product from the remainder, and bring down the next period for a new remainder. But if any of the products should be greater than the remainder, diminish the last figure of the root by one.

Double the whole root already found, for a new divisor, and continue the operation as before until all the periods are brought down.

132. If, after all the periods are brought down, there is no remainder, the given number is a perfect square.

The number of places of figures in the root will always be equal to the number of periods into which the given number is separated.

If the given number has not an exact root, there will be a remainder after all the periods are brought down; in which case ciphers may be annexed, forming new periods, for each of which there will be one decimal place in the root.

Exercises.

1. What is the square root of 36729?

	$\dot{3}\ 6\dot{7}\ 2\dot{9}$	191.64+
	1	
2 9	267	
	261	
38 1	629	
	381	
382 6	24800	
	22956	
3832 4	184400	
	153296	
	31104	

NOTE.—In this example two periods of decimals were annexed, and hence two places of decimals in the root.

2. Find the square root of 7225. *Ans.* 85.

3. Find the square root of 17689. *Ans.* 133.

4. Find the square root of 994009. *Ans.* 997.

5. Find the square root of 85673536. *Ans.* 9256.

6. Find the square root of 67798756. *Ans.* 8234.

7. Find the square root of 978121. *Ans.* 989.

8. Find the square root of 956484. *Ans.* 978.

9. What is the square root of 36372961? *Ans.* 6031.

10. What is the square root of 22071204? *Ans.* 4698.

11. What is the square root of 106929? *Ans.* 327.

12. What of 12088868379025? *Ans.* 3476905.

13. What of 2268741? *Ans.* 1506.23+.

14. What of 7596796? *Ans.* 2756.22+.

15. What is the square root of 96? *Ans.* 9.79795+.

16. What is the square root of 153? *Ans.* 12.36931+.

17. What is the square root of 101? *Ans.* 10.04987+.

18. What of 285970396644? *Ans.* 534762.

19. What of 41605800625? *Ans.* 203975.

20. What of 48303584206084? *Ans.* 6950078.

133. Since the square or second power of a fraction is obtained by squaring the numerator and denominator separately, it follows that

The square root of a fraction will be equal to the square root of the numerator divided by the square root of the denominator.

For example, the square root of $\frac{a^2}{b^2}$ is equal to $\frac{a}{b}$: for

$$\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}.$$

Exercises.

1. What is the square root of $\frac{1}{4}$? *Ans.* $\frac{1}{2}$.

2. What is the square root of $\frac{9}{16}$? *Ans.* $\frac{3}{4}$.

3. What is the square root of $\frac{64}{81}$? *Ans.* $\frac{8}{9}$.

4. What is the square root of $\frac{256}{361}$? *Ans.* $\frac{16}{19}$.

5. What is the square root of $\frac{16}{64}$? *Ans.* $\frac{1}{2}$.

6. What is the square root of $\frac{4096}{61009}$? *Ans.* $\frac{64}{247}$.

7. What is the square root of $\frac{582169}{956484}$? *Ans.* $\frac{763}{978}$.

134. If the numerator and denominator are not perfect squares, the root of the fraction cannot be exactly found. We can, however, easily find the approximate root by the following rule: —

Multiply both terms of the fraction by the denominator; then extract the square root of the numerator, and divide this root by the square root of the denominator. The quotient will be the approximate root.

ercises.

1. Find the square root of $\frac{3}{5}$. *Ans.* .7745+.

Multiplying the numerator and denominator by 5,

$$\sqrt{\frac{3}{5}} = \sqrt{\frac{15}{25}} = \frac{\sqrt{15}}{5} = (3.8729+) \div 5.$$

Hence $(3.8729+) \div 5 = .7745+.$

2. What is the square root of $\frac{7}{4}$? *Ans.* 1.32287+.

3. What is the square root of $\frac{14}{9}$? *Ans.* 1.24721+.

4. What is the square root of $11\frac{11}{16}$? *Ans.* 3.41869+.

5. What is the square root of $7\frac{13}{36}$? *Ans.* 2.71313+.

6. What is the square root of $8\frac{15}{49}$? *Ans.* 2.88203+.

7. What is the square root of $\frac{5}{12}$? *Ans.* 0.64549+.

8. What is the square root of $10\frac{3}{10}$? *Ans.* 3.20936+.

135. Finally, instead of the last method, we may, if we please,

Change the common fraction into a decimal, and continue the division until the number of decimal places is double the number of places required in the root. Then extract the root of the decimal by the last rule.

Exercises.

1. Extract the square root of $\frac{11}{14}$ to within .001.

This number, reduced to decimals, is 0.785714 to within 0.000001; but the root of 0.785714 to the nearest unit is .886. Hence 0.886 is the root of $\frac{11}{14}$ to within .001.

2. Find the $\sqrt{2\frac{13}{15}}$ to within 0.0001. *Ans.* 1.6931+.

3. What is the square root of $\frac{1}{17}$? *Ans.* 0.24253+.

4. What is the square root of $\frac{7}{8}$? *Ans.* 0.93541+.

5. What is the square root of $\frac{5}{3}$? *Ans.* 1.29099+.

136. In order to discover the process for extracting the square root of a monomial, we must see how its square is formed.

By the rule for the multiplication of monomials (§ 42), we have

$$(5a^2b^3c)^2 = 5a^2b^3c \times 5a^2b^3c = 25a^4b^6c^2;$$

that is, in order to square a monomial, it is necessary to *square its coefficient, and double the exponent of each of the letters*. Hence, to find the square root of a monomial, we have the following rule: —

Extract the square root of the coefficient, for a new coefficient.

Divide the exponent of each letter by 2, and then annex all the letters with their new exponents.

Since like signs in two factors give a plus sign in the product, the square of $-a$, as well as that of $+a$, will be $+a^2$: hence the square root of a^2 is either $+a$ or $-a$; also the square root of $25a^2b^4$ is either $+5ab^2$ or $-5ab^2$. Whence we conclude that if a monomial is positive, its square root may be affected either with the sign $+$ or $-$. Thus, $\sqrt{9a^4} = \pm 3a^2$; for $+3a^2$ or $-3a^2$, squared, gives $+9a^4$. The double sign $\pm$, with which the root is affected, is read "**plus and minus.**"

Exercises.

1. What is the square root of $64a^6b^4$?

$$\sqrt{64a^6b^4} = +8a^3b^2, \text{ for } +8a^3b^2 \times +8a^3b^2 = +64a^6b^4;$$

and $$\sqrt{64a^6b^4} = -8a^3b^2, \text{ for } -8a^3b^2 \times -8a^3b^2 = +64a^6b^4.$$

Hence $$\sqrt{64a^6b^4} = \pm 8a^3b^4.$$

2. Find the square root of $625a^2b^8c^6$. *Ans.* $\pm 25ab^4c^3$.
3. Find the square root of $576a^4b^6c^8$. *Ans.* $\pm 24a^2b^3c^4$.
4. Find the square root of $196x^6y^2z^4$. *Ans.* $\pm 14x^3yz^2$.

5. Find the square root of $441a^8b^6c^{10}d^{16}$. *Ans.* $\pm 21a^4b^3c^5d^8$.

6. Find the square root of $784a^{12}b^{14}c^{16}d^2$. *Ans.* $\pm 28a^6b^7c^8d$.

7. Find the square root of $81a^8b^4c^6$. *Ans.* $\pm 9a^4b^2c^3$.

137. From the preceding rule it follows that when a monomial is a perfect square, *its numerical coefficient is a perfect square, and all its exponents even numbers.* Thus, $25a^4b^2$ is a perfect square.

If the proposed monomial were *negative,* it would be impossible to extract its square root, since it has just been shown (§ 136) that the square of every quantity, whether positive or negative, is essentially positive. Therefore

$$\sqrt{-9}, \quad \sqrt{-4a^2}, \quad \sqrt{-8a^2b},$$

are algebraic symbols which indicate operations that cannot be performed. They are called **imaginary quantities,** or rather **imaginary expressions,** and are frequently met with in the resolution of equations of the second degree.

138. When the coefficient is not a *perfect square,* or when the exponent of any letter is *uneven,* the monomial is an *imperfect square.* Thus, $98ab^4$ is an *imperfect square.* Its root is then indicated by means of the radical sign. Thus, $\sqrt{98ab^4}$. Such quantities are called **surds of the second degree.**

Cube Root.

138 a. The **cube root** of a number is one of three equal factors of the number.

To extract the cube root of a number is to find a factor which, multiplied into itself twice, will produce the given number.

Thus, 2 is the cube root of 8, for $2 \times 2 \times 2 = 8$; and 3 is the cube root of 27, for $3 \times 3 \times 3 = 27$.

Roots . . .	1,	2,	3,	4,	5,	6,	7,	8,	9.
Cubes . . .	1,	8,	27,	64,	125,	216,	343,	512,	729.

The numbers in the first line are the cube roots of the corresponding numbers of the second. The numbers of the second line are called **perfect cubes.** By examining the numbers of the two lines, we see, first, that the cube of units cannot give a higher order than hundreds; second, that since the cube of one ten (10) is 1000, and the cube of 9 tens (90) is 81,000, *the cube of tens will not give a lower denomination than thousands, nor a higher denomination than hundreds of thousands.*

Hence, if a number contains more than three figures, its cube root will contain more than one; if it contains more than six, its root will contain more than two; and so on, every additional three figures giving one additional figure in the root; and the figures which remain at the left hand, although less than three, will also give a figure in the root. This law explains the reason for pointing off into periods of three figures each.

Let us now see how the cube of any number, as 16, is formed. Sixteen is composed of 1 ten and 6 units, and may be written $10 + 6$. To find the cube of 16, or of $10 + 6$, we must multiply the number by itself twice.

To do this we place the number thus:—		$16 =$	$10 +$	6
			$10 +$	6
Product by the units			$60 +$	36
Product by the tens		$100 +$	60	
Square of 16		$100 +$	$120 +$	36
Multiply again by 16			$10 +$	6
Product by the units		$600 +$	$720 +$	216
Product by the tens . . .	$1000 +$	$1200 +$	360	
Cube of 16	$1000 +$	$1800 +$	$1080 +$	216

By examining the parts of this number, it is seen that the first part, 1000, is the *cube of the tens;* that is,

$$10 \times 10 \times 10 = 1000.$$

The second part, 1800, is *three times the square of the tens multiplied by the units;* that is,

$$3 \times (10)^2 \times 6 = 3 \times 100 \times 6 = 1800.$$

The third part, 1080, is *three times the square of the units multiplied by the tens;* that is,

$$3 \times 6^2 \times 10 = 3 \times 36 \times 10 = 1080.$$

The fourth part is *the cube of the units;* that is,

$$6^3 = 6 \times 6 \times 6 = 216.$$

Take, for example, the following: —

What is the cube root of the number 4096?

$$\begin{array}{rl} & \dot{4}\,09\dot{6}\,(16 \\ & 1 \\ \hline 1^2 \times \underline{3} = 3) & 3\,0 \quad (9, 8, 7, 6 \\ 16^3 = & 4\,096 \end{array}$$

Since the number contains more than three figures, we know that the root will contain at least units and tens.

Separating the three right-hand figures from the 4, we know that the cube of the tens will be found in the 4, and 1 is the greatest cube in 4.

Hence we place the root 1 on the right, and this is the tens of the required root. We then cube 1, and subtract the result from 4; and to the remainder we bring down the first figure, 0, of the next period.

We have seen that the second part of the cube of 16, viz., 1800, is *three times the square of the tens multiplied by the units;* and hence it can have no significant figure of a less denomination than hundreds. It must therefore make up a part of the 30 hundreds above. But this 30 hundreds also contains all the hundreds which come from the third and fourth parts of the cube of 16. If it were not so, the 30 hundreds, divided

by three times the square of the tens, would give the unit figure exactly.

Forming a divisor of three times the square of the tens, we find the quotient to be 10; but this we know to be too large. Placing 9 in the root and cubing 19, we find the result to be 6859. Then trying 8, we find the cube of 18 still too large; but when we take 6, we find the exact number. Hence the cube root of 4096 is 16.

Hence, to find the cube root of a number,

Separate the given number into periods of three figures each, by placing a dot over the place of units, a second over the place of thousands, and so on over each third figure to the left. The left-hand period will often contain less than three places of figures.

Note the greatest perfect cube in the first period, and set its root on the right after the manner of a quotient in division. Subtract the cube of this number from the first period, and to the remainder bring down the first figure of the next period for a dividend.

Take three times the square of the root just found for a trial divisor, and see how often it is contained in the dividend, and place the quotient for a second figure of the root. Then cube the figures of the root thus found, and, if their cube be greater than the first two periods of the given number, diminish the last figure; but if it be less, subtract it from the first two periods, and to the remainder bring down the first figure of the next period for a new dividend.

Take three times the square of the whole root for a second trial divisor, and find a third figure of the root. Cube the whole root thus found, and subtract the result from the first three periods of the given number when it is less than that number, but if it is greater, diminish the figure of the root. Proceed in a similar way for all the periods.

Take the following example : —

What is the cube root of 99252847 ?

$$
\begin{array}{rll}
 & \dot{9}\dot{9}\ 25\dot{2}\ 84\dot{7}\ \underline{|463} & \\
4^3 = & 64 & \\
4^2 \times 3 = 48 & \overline{)352} & \text{1st dividend.} \\
\text{First two periods} & \dot{9}\dot{9}\ 25\dot{2} & \\
(46)^3 = 46 \times 46 \times 46 = & \underline{97\ 336} & \\
3 \times (46)^2 = 6348 & \overline{)1\ 9168} & \text{2d dividend.} \\
\text{First three periods} & \dot{9}\dot{9}\ 25\dot{2}\ 84\dot{7} & \\
(463)^3 = & 99\ 252\ 847 &
\end{array}
$$

Exercises.

Find the cube roots of the following numbers : —

1. Of 389017.
2. Of 5735339.
3. Of 32461759.
4. Of 84604519.
5. Of 259694072.
6. Of 48228544.

138 b. To extract the cube root of a decimal fraction,

Annex ciphers to the decimal, if necessary, so that it shall consist of three, six, nine, etc., places. Then put the first point over the place of thousandths, the second over the place of millionths, and so on over every third place to the right; after which extract the root as in whole numbers.

NOTES. — 1. There will be as many decimal places in the root as there are periods in the given number.

2. The same rule applies when the given number is composed of a whole number and a decimal.

3. If in extracting the root of a number there is a remainder after all the periods have been brought down, periods of ciphers may be annexed by considering them as decimals.

Exercises.

Find the cube roots of the following numbers: —

1. Of .157464.
2. Of .870983875.
3. Of 12.977875.
4. Of .751089429.
5. Of .353393243.
6. Of 3.408862625.

138 c. To extract the cube root of a common fraction,

Reduce compound fractions to simple ones, mixed numbers to improper fractions, and then reduce the fraction to its lowest terms.

Extract the cube root of numerator and denominator separately if they have exact roots; but, if either of them has not an exact root, reduce the fraction to a decimal, and extract the root as in the last case.

Exercises.

Find the cube roots of the following fractions: —

1. Of $\frac{250}{686}$.
2. Of $31\frac{15}{343}$.
3. Of $\frac{324}{1500}$.
4. Of $\frac{4}{7}$.
5. Of $\frac{5}{9}$.
6. Of $\frac{2}{3}$.

Any Root.

138 d. To find a rule for extracting *any* root of a monomial, we have simply to reverse the rule of § 119 for finding any power of a monomial. This gives for the extraction of any root of a monomial the following rule: —

Extract the required root of the coefficient, for a new coefficient.

Divide the exponent of each letter by the index of the root.

Exercises.

Extract the indicated roots of the following monomials: —

1. $\sqrt[3]{64a^9b^3c^6}$. *Ans.* $+4a^3bc^2$.
2. $\sqrt[4]{16a^8b^{12}c^4}$. *Ans.* $\pm 2a^2b^3c$.

NOTE. — It may be shown, as in § 119, that any power of an even degree of a quantity is $+$: hence every root of an even degree must have the double sign $\pm$. Every power of an odd degree of a quantity has the same sign as the quantity: hence every root of an odd degree of a quantity must have the same sign as the quantity.

3. $\sqrt[3]{-8a^3}$. *Ans.* $-2a$.
4. $\sqrt[5]{-32a^{10}b^5}$. *Ans.* $-2a^2b$.
5. $\sqrt[4]{81a^4b^{12}}$. *Ans.* $\pm 3ab^3$.
6. $\sqrt[3]{8a^6b^3c^{12}}$. *Ans.* $+2a^2bc^4$.
7. $\sqrt[4]{81a^4b^8c^{16}}$. *Ans.* $\pm 3ab^2c^4$.
8. $\sqrt[5]{-32a^5c^{10}d^{15}}$. *Ans.* $-2ac^2d^3$.
9. $\sqrt[3]{-125a^9b^6c^3}$. *Ans.* $-5a^3b^2c$.

Any root of an *even* degree of a negative quantity is imaginary; for there is no quantity which, raised to a power of an even degree, will give a negative result. Thus,

$$\sqrt[4]{-16a^4}, \quad \sqrt[6]{-64x^6y^{12}},$$

are imaginary quantities.

When the coefficient is not a perfect power of the degree indicated, or when the exponent of any letter is not exactly divisible by the index of the root, the monomial is an imperfect power of the degree indicated. Thus, $16a^2b^4$ is an imperfect cube, and $21xy^3$ is an imperfect fourth power. Their roots are indicated thus: —

$$\sqrt[3]{16a^2b^4}, \quad \sqrt[4]{21xy^3}.$$

Such quantities are called **surds.** Hence a **surd** is the indicated root of an imperfect power of the degree indicated.

138 e. For the rule to extract any root of a fraction, reverse the rule given in § 120 for finding any power of a fraction, and we have the rule: —

Extract the required root of the numerator, for a new numerator; and the required root of the denominator, for a new denominator.

Exercises.

Find the indicated roots of the following fractions: —

1. $\sqrt[3]{\dfrac{8a^3y^3}{27d^6z^6}}$. *Ans.* $\dfrac{2ay}{3d^2z^2}$.

2. $\sqrt[4]{\dfrac{16a^4y^4}{625z^8}}$. *Ans.* $\pm\dfrac{2ay}{5z^2}$.

3. $\sqrt[3]{\dfrac{343x^{-3}y^{-6}}{a^{-9}b^{-18}c^6}}$. *Ans.* $\dfrac{7x^{-1}y^{-2}}{a^{-3}b^{-6}c^2}$.

4. $\sqrt[4]{\dfrac{a^4b^8c^4}{16x^8y^8}}$. *Ans.* $\pm\dfrac{ab^2c}{2x^2y^2}$.

5. $\sqrt[5]{-\dfrac{b^5c^{5n}x^{5m}}{32y^{5p}z^{5q}}}$. *Ans.* $-\dfrac{bc^nx^m}{2y^pz^q}$.

6. $\sqrt[7]{-\dfrac{a^7x^7y^{21}}{128b^7z^{14}}}$. *Ans.* $-\dfrac{axy^3}{2bz^2}$.

RADICALS.

Transformation of Radicals.

139. Let a and b denote any two numbers, and p the product of their square roots. Then

$$\sqrt{a} \times \sqrt{b} = p \qquad (1)$$

Squaring both members, we have

$$a \times b = p^2 \qquad (2)$$

Extracting the square root of both members of (2), we have

$$\sqrt{ab} = p \qquad (3)$$

Since the second members are the same in Equations (1) and (3), the first members are equal; that is,

The square root of the product of two quantities is equal to the product of their square roots.

Similarly, if n denotes *any* number, then $n\sqrt{a}$ will denote the nth root; that is, *any* root of a. Let l denote the products of the nth roots of a and b. Then

$$\sqrt[n]{a} \times \sqrt[n]{b} = l \qquad (1)$$

Raising both members to the nth power, we have

$$a \times b = l^n \qquad (2)$$

Extracting the nth root of both members of (2), we have

$$\sqrt[n]{ab} = l \qquad (3)$$

Since the second members are the same in Equations (1) and (3), the first members are equal; that is,

The n*th root of the product of two quantities is equal to the product of their* n*th roots.*

140. Let a and b denote any two numbers, and q the quotient of their square roots. Then

$$\frac{\sqrt{a}}{\sqrt{b}} = q \qquad (1)$$

Squaring both members, we have

$$\frac{a}{b} = q^2 \qquad (2)$$

Extracting the square root of both members of (2), we have

$$\sqrt{\frac{a}{b}} = q \qquad (3)$$

Since the second members are the same in Equations (1) and (3), the first members are equal; that is,

The square root of the quotient of two quantities is equal to the quotient of their square roots.

Similarly, if r denotes the quotient of the nth roots of a and b, we have

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = r \qquad (1)$$

Raising both members to the nth power, we have

$$\frac{a}{b} = r^n \qquad (2)$$

Extracting the nth root of both members of (2), we have

$$\sqrt[n]{\frac{a}{b}} = r \qquad (3)$$

Since the second members are the same in Equations (1) and (3), the first members are equal; that is,

The n*th root of the quotient of two quantities is equal to the quotient of their* n*th roots.*

These principles enable us to *transform* radical expressions, or to *reduce* them to simpler forms. Thus, the expression

$$98ab^4 = 49b^4 \times 2a.$$

Hence $$\sqrt{98ab^4} = \sqrt{49b^4 \times 2a};$$

and, by the principle of § 139,

$$\sqrt{49b^4 \times 2a} = \sqrt{49b^4} \times \sqrt{2a} = 7b^2\sqrt{2a}.$$

In like manner,

$$\sqrt{45a^2b^3c^2d} = \sqrt{9a^2b^2c^2 \times 5bd} = 3abc\sqrt{5bd}.$$

$$\sqrt{864a^2b^5c^{11}} = \sqrt{144a^2b^4c^{10} \times 6bc} = 12ab^2c^5\sqrt{6bc}.$$

Similarly,

$$\sqrt[3]{54x^3y^7z} = \sqrt[3]{27x^3y^6 \times 2yz} = 3xy^2\sqrt[3]{2yz}.$$

The **coefficient** of a radical is the quantity without the sign. Thus, in the expressions

$$7b^2\sqrt{2a}, \quad 3abc\sqrt[3]{5bd}, \quad 12ab^2c^5\sqrt[4]{6bc},$$

the quantities $7b^2$, $3abc$, $12ab^2c^5$, are *coefficients of the radicals.*

141. Hence, to simplify a radical of the second degree, we have the following rule: —

Resolve the expression under the radical sign into two factors, one of which shall be a perfect square.

Extract the square root of the perfect square, and then multiply this root by the indicated square root of the remaining factor.

In like manner, to simplify a radical of the nth degree, that is, of *any* degree, we have the rule: —

Resolve the expression under the radical sign into two factors, one of which shall be a perfect nth *power.*

Extract the nth *root of the perfect* nth *power, and then multiply this root by the indicated* nth *root of the remaining factor.*

Exercises.

Reduce the following radicals to their simplest form: —

1. $\sqrt{75a^3bc}$. *Ans.* $5a\sqrt{3abc}$.

2. $\sqrt{128b^5a^6d^2}$. *Ans.* $8b^2a^3d\sqrt{2b}$.

3. $\sqrt{32a^9b^8c}$. *Ans.* $4a^4b^4\sqrt{2ac}$.

4. $\sqrt{256a^2b^4c^8}$. *Ans.* $16ab^2c^4$.

5. $\sqrt{1024a^9b^7c^5}$. *Ans.* $32a^4b^3c^2\sqrt{abc}$.

6. $\sqrt{729a^7b^5c^6d}$. *Ans.* $27a^3b^2c^3\sqrt{abd}$.

7. $\sqrt{675a^7b^5c^2d}$. *Ans.* $15a^3b^2c\sqrt{3abd}$.

8. $\sqrt{1445a^3c^8d^4}$. *Ans.* $17ac^4d^2\sqrt{5a}$.

9. $\sqrt{1008a^9d^7m^8}$. *Ans.* $12a^4d^3m^4\sqrt{7ad}$.

10. $\sqrt{2156a^{10}b^8c^6}$. *Ans.* $14a^5b^4c^3\sqrt{11}$.

11. $\sqrt{405a^7b^6d^8}$. *Ans.* $9a^3b^3d^4\sqrt{5a}$.

12. $\sqrt[4]{48a^5b^2c^4}$. *Ans.* $2ac\sqrt[4]{3ab^2}$.

13. $\sqrt[3]{112a^4b^6}$. *Ans.* $2ab^2\sqrt[3]{14a}$.

14. $2\sqrt[3]{54x^{-4}y^{-5}}$. *Ans.* $6x^{-1}y^{-2}\sqrt[3]{2x^{-1}}$.

15. $\sqrt[4]{512a^{4m}y^{-5}}$. *Ans.* $4a^my^{-1}\sqrt[4]{2y^{-1}}$.

16. $3\sqrt[3]{128x^5y^{-5}}$. *Ans.* $12xy^{-1}\sqrt[3]{2x^2y^{-2}}$.

17. $\sqrt[3]{-24x^{-3}y^5}$. *Ans.* $-2x^{-1}y\sqrt[3]{3y^2}$.

142. A *coefficient*, or a *factor of a coefficient*, may be carried under the radical sign, raising it to the proper power, *by squaring it* if the radical is of the second degree, by cubing it if the radical is of the third degree, by raising it to the

fourth power if the radical is of the fourth degree, and so on. Thus,

(1) $3a^2\sqrt{bc} = \sqrt{(3a^2)^2 \times bc} = \sqrt{9a^4bc}.$

(2) $2ab\sqrt{d} = 2\sqrt{a^2b^2d} = \sqrt{4a^2b^2d}.$

(3) $4(a+b)\sqrt{a-b} = 4\sqrt{(a+b)^2(a-b)} = 4\sqrt{(a^2-b^2)(a+b)}.$

(4) $5bc\sqrt{a^2-c^2} = 5\sqrt{b^2c^2(a^2-c^2)}.$

(5) $3a\sqrt[3]{2ax} = \sqrt[3]{(3a)^3 \times 2ax} = \sqrt{27a^3 \times 2ax} = \sqrt{54a^4x}.$

(6) $-4\sqrt[4]{ab^2} = \sqrt[4]{(-4)^4 \times ab^2} = \sqrt[4]{256ab^2}.$

(7) $3ab\sqrt[3]{x^2-y^2} = 3\sqrt[3]{a^3b^3(x^2-y^2)}.$

(8) $2x^2\sqrt[4]{a^4-b^4} = \sqrt[4]{16x^8(a^4-b^4)}.$

The square root of a negative quantity may be simplified. Thus,

$$\sqrt{-9} = \sqrt{9 \times -1} = \sqrt{9} \times \sqrt{-1} = 3\sqrt{-1},$$

and $\sqrt{-4a^2} = \sqrt{4a^2} \times \sqrt{-1} = 2a\sqrt{-1};$

also $\sqrt{-8a^2b} = \sqrt{4a^2 \times -2b} = 2a\sqrt{-2b}$

$= 2a\sqrt{2b} \times \sqrt{-1}$; that is,

The square root of a negative quantity is equal to the square root of the same quantity with a positive sign, multiplied into the square root of -1.

Exercises.

Reduce the following: —

1. $\sqrt{-64a^2b^2}.$ *Ans.* $8ab\sqrt{-1}.$

2. $\sqrt{-128a^4b^5}.$ *Ans.* $8a^2b^2\sqrt{2b}\sqrt{-1}.$

3. $\sqrt{-72a^5b^7c^6}.$ *Ans.* $6a^2b^3c^3\sqrt{2ab}\sqrt{-1}.$

4. $\sqrt{-48a^3bc^5}.$ *Ans.* $4ac^2\sqrt{3abc}\sqrt{-1}.$

NOTE. — Any even root of a negative quantity is equal to the same root of the quantity with a positive sign, multiplied into the same indicated root of -1.

5. $\sqrt[4]{-81x^4y^4} = \sqrt[4]{81x^4y^4} \times \sqrt[4]{-1} = 3xy\sqrt[4]{-1}.$

6. $\sqrt[6]{-32a^7b^8c} = \sqrt[6]{32a^6b^6 \times ab^2c} \times \sqrt{-1} = 2ab\sqrt[6]{ab^2c}\sqrt[6]{-1}.$

142 a. It is often desirable to change the index of a radical, and it may be done in accordance with the following principle: —

Let m and n represent any two numbers whatever, and let

$$\sqrt[m]{\sqrt[n]{a}} = s \qquad (1)$$

Raise both members of (1) to the mth power, and we have

$$\sqrt[n]{a} = s^m \qquad (2)$$

Raise both members of (2) to the nth power, and we have

$$a = s^{mn} \qquad (3)$$

Extract the mnth root of the members of (3), and we have

$$\sqrt[mn]{a} = s \qquad (4)$$

Since the second members of Equations (1) and (4) are the same, the second members are equal, and we have

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a};$$

that is,

The m*th root of the* n*th root of a quantity is equal to the* mn*th root of that quantity, and the reverse.*

Thus, $\sqrt[3]{\sqrt{5}} = \sqrt[6]{5},$

and $\sqrt[12]{9} = \sqrt[4]{\sqrt[3]{9}}.$

Take $\sqrt[6]{4a^2}.$

By this principle we have

$$\sqrt[6]{4a^2} = \sqrt[3]{\sqrt{4a^2}};$$

and since $$\sqrt{4a^2} = 2a,$$

we have $$\sqrt[6]{4a^2} = \sqrt[3]{2a}.$$

In like manner $$\sqrt[4]{36a^2b^2} = \sqrt{\sqrt{36a^2b^2}} = \sqrt{6ab},$$

and generally $$\sqrt[mn]{b^n} = \sqrt[m]{\sqrt[n]{b^n}} = \sqrt[m]{b};$$

that is, when the index of a radical can be divided by n, and the quantity under the radical sign is a perfect nth power, *we can divide its index by* n *provided we extract the* n*th root of the quantity under the sign.*

Thus, $$\sqrt[6]{27m^{12}n^6p^3} = \sqrt{3m^4n^2p};$$

$$\sqrt[9]{8x^3y^{3n}} = \sqrt[3]{2xy^n};$$

$$\sqrt[10]{169b^6c^4} = \sqrt[5]{13b^3c^2}.$$

The converse of this principle is true. For example:

$$2a = \sqrt{4a^2}.$$

Hence $$\sqrt[3]{2a} = \sqrt[3]{\sqrt{4a^2}} = \sqrt[6]{4a^2}.$$

Here the radical $\sqrt[3]{2a}$ has been changed, without altering its value, by multiplying the index by 2 and at the same time raising the quantity under the radical sign $(2a)$ to the second power. As all other examples can be treated in the same way, we have the principle: —

The index of a radical may be multiplied by any number, provided we raise the quantity under the radical sign to a power of which this number is the exponent.

Thus, $$\sqrt[3]{4a^2b} = \sqrt[6]{16a^4b^2};$$

$$\sqrt{2} = \sqrt[8]{16};$$

$$\sqrt[3]{-\frac{1}{2}} = \sqrt[9]{-\frac{1}{8}} = \sqrt[18]{\frac{1}{64}}.$$

The same principles are illustrated when the radical is denoted by a fractional exponent. Thus,

$$(a)^{\frac{1}{3}} = (a)^{\frac{3}{9}}, \quad \text{since } \frac{1}{3} = \frac{3}{9};$$

also

$$(a)^{\frac{3}{12}} = (a)^{\frac{1}{4}}, \quad \text{since } \frac{3}{12} = \frac{1}{4};$$

that is,

$$a^{\frac{1}{3}} = a^{\frac{3}{9}}, \quad \text{and} \quad a^{\frac{3}{12}} = a^{\frac{1}{4}};$$

or, by writing the equivalent expressions,

$$\sqrt[3]{a} = \sqrt[9]{a^3}, \qquad \sqrt[12]{a^3} = \sqrt[4]{a},$$

the same results as obtained before under the rules. Hence, in the case of fractional exponents, we may

Multiply or divide both terms of the fractional exponent by the same number without changing the value of the radical.

By means of these principles, radicals having different indices may be reduced to equivalent radicals having a common index. Take the radicals $\sqrt{a}$, $\sqrt[3]{c}$, and $\sqrt[4]{b}$. The L. C. M. of the indices 2, 3, and 4, is 12. Reducing each to the index 12 by the foregoing principle, we have

$$\sqrt{a} = \sqrt[12]{a^6}, \quad \sqrt[3]{c} = \sqrt[12]{c^4}, \quad \text{and} \quad \sqrt[4]{b} = \sqrt[12]{b^3}.$$

As all other cases may be treated in the same way, we have the rule: —

Find the L. C. M. of the indices, and reduce each radical to that index.

When the radicals are expressed by fractional exponents, we have simply to reduce these exponents to a common denominator. Thus, $a^{\frac{1}{2}}$, b^2, $c^{\frac{2}{3}}$, and $d^{\frac{2}{3}}$ become, when reduced to a common index 6, $a^{\frac{3}{6}}$, $b^{\frac{12}{6}}$, $c^{\frac{4}{6}}$, and $d^{\frac{4}{6}}$; or $(a^3)^{\frac{1}{6}}$, $(b^{12})^{\frac{1}{6}}$, $(c^4)^{\frac{1}{6}}$, and $(d^4)^{\frac{1}{6}}$.

In like manner reduce 2, $(3)^{\frac{1}{3}}$, $(a)^{\frac{1}{2}}$, and $(b)^{\frac{1}{4}}$ to a common index.

Ans. $(2^{12})^{\frac{1}{12}}$, $(3^4)^{\frac{1}{12}}$, $(a^6)^{\frac{1}{12}}$, $(b^3)^{\frac{1}{12}}$; or $\sqrt[12]{2}$, $\sqrt[12]{3^4}$, $\sqrt[12]{a^6}$, $\sqrt[12]{b^3}$.

Addition of Radicals.

143. Similar radicals are those which have a common unit or radical part. Thus, the radicals $3\sqrt{b}$ and $5c\sqrt{b}$ are similar, and so also are $9\sqrt{2}$ and $7\sqrt{2}$; $4\sqrt[3]{c}$, $a\sqrt[3]{c}$, and $-3\sqrt[3]{c}$ are similar, as are $\sqrt[4]{3a}$, $9\sqrt[4]{3a}$, $7c\sqrt[4]{3a}$, and $xy\sqrt[4]{3a}$.

144. Radicals are added like other algebraic quantities. Hence the following rule: —

If the radicals are similar, add their coefficients, and to the sum annex the common unit or radical.

If the radicals are not similar, connect them together with their proper signs.

Thus, $\quad 3a\sqrt{b}+5c\sqrt{b}=(3a+5c)\sqrt{b}.$

In like manner

$$7\sqrt{2a}+3\sqrt{2a}=(7+3)\sqrt{2a}=10\sqrt{2a}.$$

$$3\sqrt[3]{4b}+5\sqrt[3]{4b}=8\sqrt[3]{4b}.$$

$$a\sqrt[5]{21}+b\sqrt[5]{21}=(a+b)\sqrt[5]{21}.$$

NOTES. — 1. Two radicals, which do not appear to be similar at first sight, may become so by transformation (§§ 141, 142 *a*). Thus,

$$\sqrt{48ab^2}+b\sqrt{75a}=4b\sqrt{3a}+5b\sqrt{3a}=9b\sqrt{3a}.$$

$$2\sqrt{45}+3\sqrt{5}=6\sqrt{5}+3\sqrt{5}=9\sqrt{5}.$$

$$3\sqrt[6]{4a^2}+2\sqrt[3]{2a}=3\sqrt[3]{2a}+2\sqrt[3]{2a}=5\sqrt[3]{2a}.$$

$$5\sqrt{8}+4\sqrt[6]{512}=5\sqrt{8}+4\sqrt{8}=9\sqrt{8}.$$

2. When the radicals are not similar, the addition or subtraction can only be indicated. Thus, in order to add $3\sqrt{b}$ to $5\sqrt{a}$, we write $5\sqrt{a}+3\sqrt{b}$; and to add $2\sqrt[3]{c}$ and $3\sqrt[4]{c}$, we write

$$2\sqrt[3]{c}+3\sqrt[4]{c}, \text{ or } 2\sqrt[12]{c^4}+3\sqrt[12]{c^3}.$$

Exercises.

Add the following : —

1. $\sqrt{27a^2}$ and $\sqrt{48a^2}$. *Ans.* $7a\sqrt{3}$.

2. $\sqrt{50a^4b^2}$ and $\sqrt{72a^4b^2}$. *Ans.* $11a^2b\sqrt{2}$.

3. $\sqrt{\frac{3a^2}{5}}$ and $\sqrt{\frac{a^2}{15}}$. *Ans.* $4a\sqrt{\frac{1}{15}}$.

4. $\sqrt{125}$ and $\sqrt{500a^2}$. *Ans.* $(5+10a)\sqrt{5}$.

5. $\sqrt{\frac{50}{147}}$ and $\sqrt{\frac{100}{294}}$. *Ans.* $\frac{10}{21}\sqrt{6}$.

6. $\sqrt{98a^2x}$ and $\sqrt{36x^2-36a^2}$. *Ans.* $7a\sqrt{2x}+6\sqrt{x^2-a^2}$.

7. $\sqrt{98a^2x}$ and $\sqrt{288a^4x^5}$. *Ans.* $(7a+12a^2x^2)\sqrt{2x}$.

8. $\sqrt{72}$ and $\sqrt{128}$. *Ans.* $14\sqrt{2}$.

9. $\sqrt{27}$ and $\sqrt{147}$. *Ans.* $10\sqrt{3}$.

10. $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{27}{50}}$. *Ans.* $\frac{19}{30}\sqrt{6}$.

11. $2\sqrt{a^2b}$ and $3\sqrt{64bx^4}$. *Ans.* $(2a+24x^2)\sqrt{b}$.

12. $\sqrt{243}$ and $10\sqrt{363}$. *Ans.* $119\sqrt{3}$.

13. $\sqrt{320a^2b^2}$ and $\sqrt{245a^8b^6}$. *Ans.* $(8ab+7a^4b^3)\sqrt{5}$.

14. $\sqrt{75a^6b^7}$ and $\sqrt{300a^6b^5}$. *Ans.* $(5a^3b^3+10a^3b^2)\sqrt{3b}$.

15. $3b\sqrt[3]{2a^5b^2}$, $7\sqrt[3]{2a^5b^5}$, and $8a\sqrt[3]{2a^2b^5}$. *Ans.* $18ab\sqrt[3]{2a^2b^2}$.

16. $\sqrt[3]{8a^3b+16a^4}$ and $\sqrt[3]{b^4+2ab^3}$. *Ans.* $(2a+b)\sqrt[3]{2a+b}$.

17. $6\sqrt[6]{4a^2}$, $2\sqrt[3]{2a}$, and $\sqrt[9]{8a^3}$. *Ans.* $9\sqrt[3]{2a}$.

18. $12\sqrt[3]{\frac{1}{4}}$ and $3\sqrt[3]{\frac{1}{32}}$.

$$\sqrt[3]{\frac{1}{4}}=\sqrt[3]{\frac{2}{8}}=\sqrt[3]{2}\times\frac{1}{8}=\frac{1}{2}\sqrt[3]{2}.$$

Hence $$12\sqrt[3]{\frac{1}{4}}=\frac{12}{2}\sqrt[3]{2}=6\sqrt[3]{2},$$

and $$\sqrt[3]{\frac{1}{32}}=\sqrt[3]{\frac{2}{64}}=\sqrt[3]{2\times\frac{1}{64}}=\frac{1}{4}\sqrt[3]{2}.$$

Hence $$3\sqrt[3]{\frac{1}{32}}=\frac{3}{4}\sqrt[3]{2}.$$

$$\therefore\ 12\sqrt[3]{\frac{1}{4}}+3\sqrt[3]{\frac{1}{32}}=6\sqrt[3]{2}+\frac{3}{4}\sqrt[3]{2}=6\tfrac{3}{4}\sqrt[3]{2}=\frac{27}{4}\sqrt[3]{2}.$$

19. $\frac{2}{3}\sqrt[3]{\frac{2}{9}}$, $\frac{1}{6}\sqrt[3]{\frac{1}{36}}$, and $\frac{3}{5}\sqrt[3]{\frac{3}{32}}$. *Ans.* $\frac{2}{5}\sqrt[3]{6}$.

Subtraction of Radicals.

145. Radicals are subtracted like other algebraic quantities. Hence the following rule: —

If the radicals are similar, subtract the coefficient of the subtrahend from that of the minuend, and to the difference annex the common unit or radical.

If the radicals are not similar, indicate the operation by the minus sign.

Exercises.

1. What is the difference between $3a\sqrt{b}$ and $a\sqrt{b}$?

Here $$3a\sqrt{b}-a\sqrt{b}=2a\sqrt{b}.\quad \textit{Ans.}$$

2. From $9a\sqrt{27b^2}$ subtract $6a\sqrt{27b^2}$. *Ans.* $9ab\sqrt{3}$.

$$9a\sqrt{27b^2}=27ab\sqrt{3},\text{ and }6a\sqrt{27b^2}=18ab\sqrt{3}.$$
$$27ab\sqrt{3}-18ab\sqrt{3}=9ab\sqrt{3}.$$

Find the differences between the following: —

3. $\sqrt{75}$ and $\sqrt{48}$. *Ans.* $\sqrt{3}$.

4. $\sqrt{24a^2b^2}$ and $\sqrt{54b^4}$. *Ans.* $(2ab - 3b^2)\sqrt{6}$.

5. $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$. *Ans.* $\frac{4}{45}\sqrt{15}$.

6. $\sqrt{128a^3b^2}$ and $\sqrt{32a^9}$. *Ans.* $(8ab - 4a^4)\sqrt{2a}$.

7. $\sqrt{48a^3b^3}$ and $\sqrt{9ab}$. *Ans.* $4ab\sqrt{3ab} - 3\sqrt{ab}$.

8. $\sqrt{242a^5b^5}$ and $\sqrt{2a^3b^3}$. *Ans.* $(11a^2b^2 - ab)\sqrt{2ab}$.

9. $\sqrt{\frac{3}{4}}$ and $\sqrt{\frac{3}{9}}$. *Ans.* $\frac{1}{6}\sqrt{3}$.

10. $\sqrt{320a^2}$ and $\sqrt{80a^2}$. *Ans.* $4a\sqrt{5}$.

11. $\sqrt{720a^3b^3}$ and $\sqrt{245abc^2d}$. *Ans.* $(12ab - 7cd)\sqrt{5ab}$.

12. $\sqrt{968a^2b^2}$ and $\sqrt{200a^2b^2}$. *Ans.* $12ab\sqrt{2}$.

13. $\sqrt{112a^8b^6}$ and $\sqrt{28a^8b^6}$. *Ans.* $2a^4b^3\sqrt{7}$.

Note. — Two radicals which do not appear at first to be similar may become so by transformation, as in the preceding case. For example:

$$\sqrt[3]{216a^6b^4} - b\sqrt[3]{27a^6b} = \sqrt[3]{216a^6b^3 \times b} - b\sqrt[3]{27a^6 \times b}$$
$$= 6a^2b\sqrt[3]{b} - 3a^2b\sqrt[3]{b} = 3a^2b\sqrt[3]{b},$$

and $\sqrt{320} - \sqrt{80} = 8\sqrt{5} - 4\sqrt{5} = 4\sqrt{5}$.

14. From $5\sqrt[6]{4a^2}$ subtract $3\sqrt[3]{2a}$. *Ans.* $2\sqrt[3]{2a}$.

15. From $\sqrt[3]{81} + \sqrt[3]{192}$ subtract $\sqrt[3]{512}$. *Ans.* $7\sqrt[3]{3} - 8$.

Note. — Here $\sqrt[3]{512}$ is not similar to the sum of $\sqrt[3]{81}$ and $\sqrt[3]{192}$; hence the operation of subtraction can only be indicated.

16. From $\frac{2}{3}\sqrt[3]{\frac{2}{9}} + \frac{3}{5}\sqrt[3]{\frac{3}{32}}$ subtract $\frac{1}{6}\sqrt[3]{\frac{1}{36}}$. *Ans.* $\frac{31}{90}\sqrt[3]{6}$.

Multiplication of Radicals.

146. Radicals are multiplied like other algebraic quantities. Hence we have the following rule: —

Multiply the coefficients together, for a new coefficient.
Multiply the radical parts together, for a new radical part.
Then reduce the result to its simplest form.

Exercises.

1. Multiply $3a\sqrt{bc}$ by $2\sqrt{ab}$.

$$3a\sqrt{bc} \times 2\sqrt{ab} = 3a \times 2 \times \sqrt{bc} \times \sqrt{ab},$$

which, by § 139, $\quad = 6a\sqrt{b^2ac} = 6ab\sqrt{ac}$.

Multiply the following: —

2. $3\sqrt{5ab}$ and $4\sqrt{20a}$. *Ans.* $120a\sqrt{b}$.

3. $2a\sqrt{bc}$ and $3a\sqrt{bc}$. *Ans.* $6a^2bc$.

4. $2a\sqrt{a^2+b^2}$ and $-3a\sqrt{a^2+b^2}$. *Ans.* $-6a^2(a^2+b^2)$.

5. $2ab\sqrt{a+b}$ and $ac\sqrt{a-b}$. *Ans.* $2a^2bc\sqrt{a^2-b^2}$.

6. $3\sqrt{2}$ and $2\sqrt{8}$. *Ans.* 24.

7. $\frac{5}{3}\sqrt{\frac{3}{8}a^2b}$ and $\frac{2}{10}\sqrt{\frac{2}{5}c^2b}$. *Ans.* $\frac{1}{30}abc\sqrt{15}$.

8. $2x+\sqrt{b}$ and $2x-\sqrt{b}$. *Ans.* $4x^2-b$.

9. $\sqrt{a+2\sqrt{b}}$ and $\sqrt{a-2\sqrt{b}}$. *Ans.* $\sqrt{a^2-4b}$.

10. $3a\sqrt{27a^3}$ and $\sqrt{2a}$. *Ans.* $9a^3\sqrt{6}$.

NOTE. — When the radicals have not the same index, they must be reduced to a common index before multiplying. For example: to multiply $2a\sqrt{x}$ by $3\sqrt[3]{y}$, we must reduce them to the common index 6, making them $2a\sqrt[6]{x^3}$ and $3\sqrt[6]{y^2}$; and the product is $6a\sqrt[6]{x^3y^2}$.

11. $\sqrt{8}$ and $\sqrt[3]{5}$. *Ans.* $\sqrt[6]{512}\times\sqrt[6]{25}=\sqrt[6]{12800}=2\sqrt[6]{200}$.

12. $3a\sqrt[6]{b}$ and $5b\sqrt[8]{2c}$. *Ans.* $15ab\sqrt[24]{8b^4c^3}$.

13. $3a\sqrt[4]{8a^2}$ and $2b\sqrt[4]{4a^2c}$. *Ans.* $6ab\sqrt[4]{32a^4c}=12a^2b\sqrt[4]{2c}$.

NOTE. — By combining the foregoing method with the method for the multiplication of polynomials (§ 43), complicated radical expressions may be multiplied together.

14. $\sqrt[3]{x}+2\sqrt[6]{x}+4$ and $\sqrt[3]{x}+2\sqrt[6]{x}$.

First Method.

$$\begin{array}{r}
\sqrt[3]{x}+2\sqrt[6]{x}+4\phantom{+8\sqrt[6]{x}}\\
\sqrt[3]{x}+2\sqrt[6]{x}\phantom{+4+8\sqrt[6]{x}}\\
\hline
\sqrt[3]{x^2}+2\sqrt[6]{x^3}+4\sqrt[3]{x}\phantom{+8\sqrt[6]{x}}\\
+2\sqrt[6]{x^3}+4\sqrt[3]{x}+8\sqrt[6]{x}\\
\hline
\sqrt[3]{x^2}+4\sqrt{x}+8\sqrt[3]{x}+8\sqrt[6]{x}
\end{array}$$

Second Method.

$$\begin{array}{r}
x^{\frac13}+2x^{\frac16}+4\phantom{+8x^{\frac16}}\\
x^{\frac13}+2x^{\frac16}\phantom{+4+8x^{\frac16}}\\
\hline
x^{\frac23}+2x^{\frac36}+4x^{\frac13}\phantom{+8x^{\frac16}}\\
+2x^{\frac36}+4x^{\frac26}+8x^{\frac16}\\
\hline
x^{\frac23}+4x^{\frac12}+8x^{\frac13}+8x^{\frac16}
\end{array}$$

NOTE. — The two results are identical, but the second has been obtained by following the ordinary rule for exponents. Hence we conclude that the rule for multiplication is the same whether the exponents are entire or fractional.

15. $(a^{\frac34}+a^{\frac12}x^{\frac12}+a^{\frac14}x+x^{\frac32})\times(a^{\frac14}-x^{\frac12})$. *Ans.* $a-x^2$.

16. $\left(x+\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}\right)\times\left(x+\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}\right)$.

Ans. x^2+px-q.

17. $\left(\frac{x}{b}\sqrt{\frac{a}{b}}+\sqrt{\frac{c}{d}}\right)\times\left(\frac{x}{b}\sqrt{\frac{a}{b}}-\sqrt{\frac{c}{d}}\right)$. *Ans.* $\frac{ax^2}{b^3}-\frac{c}{d}$.

18. $(\sqrt[3]{a^{-\frac12}}+\sqrt[6]{a^{\frac12}b})\times(\sqrt[3]{a^{-\frac12}}-\sqrt[6]{a^{\frac12}b})$. *Ans.* $\sqrt[3]{a^{-1}}-\sqrt[3]{a^{\frac12}b}$.

19. $\left(\frac{c}{2}+\frac{1}{2}\sqrt{a^2-c^2}\right)\times\left(\frac{c}{2}+\frac{1}{2}\sqrt{a^2-c^2}\right)$.

Ans. $\frac{a^2}{4}+\frac{c}{2}\sqrt{a^2-c^2}$.

Division of Radicals.

147. Radical quantities are divided like other algebraic quantities. Hence we have the following rule: —

Divide the coefficient of the dividend by the coefficient of the divisor, for a new coefficient.

Divide the radical part of the dividend by the radical part of the divisor, for a new radical part.

Then reduce the result to its simplest form.

Exercises.

1. Divide $8a\sqrt{b^3c}$ by $4a\sqrt{bc^3}$.

$$\frac{8a}{4a} = 2, \text{ New coefficient.}$$

$$\frac{\sqrt{b^3c}}{\sqrt{bc^3}} = \sqrt{\frac{b^3c}{bc^3}} = \sqrt{\frac{b^2}{c^2}} = \frac{b}{c}. \qquad (§\ 140.)$$

Hence $$2 \times \frac{b}{c} = \frac{2b}{c}, \text{ Quotient.}$$

2. Divide $5a\sqrt{b}$ by $2b\sqrt{c}$. *Ans.* $\frac{5a}{2b}\sqrt{\frac{b}{c}}$.

3. Divide $12ac\sqrt{6bc}$ by $4c\sqrt{2b}$. *Ans.* $3a\sqrt{3c}$.

4. Divide $6a\sqrt{96b^4}$ by $3\sqrt{8b^2}$. *Ans.* $4ab\sqrt{3}$.

5. Divide $4a^2\sqrt{50b^5}$ by $2a^2\sqrt{5b}$. *Ans.* $2b^2\sqrt{10}$.

6. Divide $26a^3b\sqrt{81a^2b^2}$ by $13a\sqrt{9ab}$. *Ans.* $6a^2b\sqrt{ab}$.

7. Divide $84a^3b^4\sqrt{27ac}$ by $42ab\sqrt{3a}$. *Ans.* $6a^2b^3\sqrt{c}$.

8. Divide $\sqrt{\frac{1}{3}a^2}$ by $\sqrt{2}$. *Ans.* $\frac{1}{4}a$.

9. Divide $6a^2b^2\sqrt{20a^3}$ by $12\sqrt{5a}$. *Ans.* a^3b^2.

10. Divide $6a\sqrt{10b^2}$ by $3\sqrt{5}$. *Ans.* $2ab\sqrt{2}$.

11. Divide $48b^4\sqrt{15}$ by $2b^2\sqrt{\frac{1}{15}}$. *Ans.* $360b^2$.

12. Divide $8a^2b^4c^3\sqrt{7d^3}$ by $2a\sqrt{28d}$. *Ans.* $2ab^4c^3d$.

13. Divide $96a^4c^3\sqrt{98b^5}$ by $48abc\sqrt{2b}$. *Ans.* $14a^3bc^2$.

14. Divide $27a^5b^6\sqrt{21a^3}$ by $\sqrt{7a}$. *Ans.* $27a^6b^6\sqrt{3}$.

15. Divide $18a^8b^6\sqrt{8a^4}$ by $6ab\sqrt{a^2}$. *Ans.* $6a^8b^5\sqrt{2}$.

NOTE. — As in multiplication, if the radicals have not the same index, they must be reduced to a common index before division.

16. Divide $2\sqrt{2ax}$ by $\sqrt[3]{4bx^2}$.

$$2\sqrt{2ax} = 2\sqrt[6]{8a^3x^3},$$

and

$$\sqrt[3]{4bx^2} = \sqrt[6]{16b^2x^4}.$$

Hence

$$\frac{2\sqrt{2ax}}{\sqrt[3]{4bx^2}} = \frac{2\sqrt[6]{8a^3x^3}}{\sqrt[6]{16b^2x^4}} = 2\sqrt[6]{\frac{8a^3x^3}{16b^2x^4}} = 2\sqrt[6]{\frac{a^3}{2b^2x}}.$$

17. Divide $\frac{1}{2}\sqrt[4]{2}$ by $\sqrt[3]{3}$. *Ans.* $\frac{1}{2}\sqrt[12]{\frac{8}{81}}$.

18. Divide $2\sqrt{3a}$ by $\sqrt[3]{4ab}$. *Ans.* $2\sqrt[6]{\frac{27a}{16b^2}}$.

NOTE. — By combining the above methods with the method for the division of polynomials (§ 52), any complicated radical expression may be divided by another.

19. Divide $x + \sqrt{xy} + y$ by $\sqrt{x} + \sqrt[4]{xy} + \sqrt{y}$.

First Method.

$$\begin{array}{l|l}
x + \sqrt{xy} + y & \sqrt{x} + \sqrt[4]{xy} + \sqrt{y} \\
\underline{x + \sqrt[4]{x^3y} + \sqrt{xy}} & \overline{\sqrt{x} - \sqrt[4]{xy} + \sqrt{y}} \\
\quad - \sqrt[4]{x^3y} + y & \\
\quad \underline{- \sqrt[4]{x^3y} - \sqrt{xy} - \sqrt[4]{xy^3}} & \\
\qquad\qquad \sqrt{xy} + \sqrt[4]{xy^3} + y & \\
\qquad\qquad \underline{\sqrt{xy} + \sqrt[4]{xy^3} + y} & \\
\qquad\qquad\qquad 0 &
\end{array}$$

Second Method.

$$
\begin{array}{l|l}
x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y & \underline{x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}} \\
\underline{x + x^{\frac{3}{4}}y^{\frac{1}{4}} + x^{\frac{1}{2}}y^{\frac{1}{2}}} & x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}} \\
\quad - x^{\frac{3}{4}}y^{\frac{1}{4}} + y & \\
\quad \underline{- x^{\frac{3}{4}}y^{\frac{1}{4}} - x^{\frac{1}{2}}y^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{3}{4}}} & \\
\qquad\qquad x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{3}{4}} + y & \\
\qquad\qquad \underline{x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{3}{4}} + y} & \\
\qquad\qquad\qquad\qquad 0 &
\end{array}
$$

NOTE. — The two results are identical, but the second one has been obtained by following the ordinary rule for exponents: hence we conclude that the operation for division is the same, whether the exponents are entire or fractional.

20. $\left(16x - \frac{y^4}{16}\right) \div \left(2x^{\frac{1}{4}} - \frac{y}{2}\right)$.

Ans. $8x^{\frac{3}{4}} + 2x^{\frac{1}{2}}y + \frac{1}{2}x^{\frac{1}{4}}y^2 + \frac{1}{8}y^3$.

SQUARE ROOT OF POLYNOMIALS.

148. Before explaining the rule for the extraction of the square root of a polynomial, let us first examine the squares of several polynomials. We have

$$(a+b)^2 = a^2 + 2ab + b^2,$$

$$(a+b+c)^2 = a^2 + 2ab + b^2 + 2(a+b)c + c^2,$$

$$(a+b+c+d)^2 = a^2 + 2ab + b^2 + 2(a+b)c + c^2 + 2(a+b+c)d + d^2.$$

The *law* by which these squares are formed can be enumerated thus: —

The square of any polynomial is equal to the square of the first term, plus twice the product of the first term by the second,

plus the square of the second; plus twice the sum of the first two terms multiplied by the third, plus the square of the third; plus twice the sum of the first three terms multiplied by the fourth, plus the square of the fourth; and so on.

149. Hence, to extract the square root of a polynomial, we have the following rule: —

Arrange the polynomial with reference to one of its letters, and extract the square root of the first term. This will give the first term of the root.

Divide the second term of the polynomial by double the first term of the root, and the quotient will be the second term of the root.

Then form the square of the algebraic sum of the two terms of the root found, and subtract it from the first polynomial; and then divide the first term of the remainder by double the first term of the root; and the quotient will be the third term.

Form the double product of the sum of the first and second terms by the third, and add the square of the third; then subtract this result from the last remainder, and divide the first term of the result so obtained by double the first term of the root; and the quotient will be the fourth term.

Then proceed in a similar manner to find the other terms.

Exercises.

1. Extract the square root of the polynomial

$$49a^2b^2 - 24ab^3 + 25a^4 - 30a^3b + 16b^4.$$

First arrange it with reference to the letter a.

$$\begin{array}{rl|l}
25a^4 - 30a^3b + 49a^2b^2 - 24ab^3 + 16b^4 & & 5a^2 - 3ab + 4b^2 \\
\underline{25a^4 - 30a^3b + 9a^2b^2 } & & 10a^2 \\
40a^2b^2 - 24ab^3 + 16b^4 & & \text{1st rem.} \\
\underline{40a^2b^2 - 24ab^3 + 16b^4} & & \\
0 & & \text{2d rem.}
\end{array}$$

After having arranged the polynomial with reference to a, extract the square root of $25a^4$: this gives $5a^2$, which is placed at the right of the polynomial. Then divide the second term, $-30a^3b$, by the double of $5a^2$, or $10a^2$: the quotient is $-3ab$, which is placed at the right of $5a^2$. Hence the first two terms of the root are $5a^2-3ab$. Squaring this binomial, it becomes $25a^4-30a^3b+9a^2b^2$, which, subtracted from the given polynomial, leaves a remainder of which the first term is $40a^2b^2$. Dividing this first term by $10a^2$ (the double of $5a^2$), the quotient is $+4b^2$: this is the third term of the root, and is written on the right of the first two terms. By forming the double product of $5a^2-3ab$ by $4b^2$, squaring $4b^2$, and taking the sum, we find the polynomial $40a^2b^2-24ab^3+16b^4$, which, subtracted from the first remainder, gives 0. Therefore $5a^2-3ab+4b^2$ is the required root.

If a final remainder is found equal to 0, the root is exact; if not, the root is only approximate.

2. Find the square root of $a^4+4a^3x+6a^2x^2+4ax^3+x^4$.
Ans. $a^2+2ax+x^2$.

3. Find the square root of $a^4-4a^3x+6a^2x^2-4ax^3+x^4$.
Ans. $a^2-2ax+x^2$.

4. Find the square root of $4x^6+12x^5+5x^4-2x^3+7x^2-2x+1$.
Ans. $2x^3+3x^2-x+1$.

5. Find the square root of $9a^4-12a^3b+28a^2b^2-16ab^3+16b^4$.
Ans. $3a^2-2ab+4b^2$.

6. What is the square root of $x^4-4ax^3+4a^2x^2-4x^2+8ax+4$?
Ans. $x^2-2ax-2$.

7. What is the square root of $9x^2-12x+6xy+y^2-4y+4$?
Ans. $3x+y-2$.

8. What is the square root of $y^4-2y^2x^2+2x^2-2y^2+1+x^4$?
Ans. y^2-x^2-1.

9. What is the square root of $9a^4b^4-30a^3b^3+25a^2b^2$?
Ans. $3a^2b^2-5ab$.

10. Find the square root of $25a^4b^2 - 40a^3b^2c + 76a^2b^2c^2 - 48ab^2c^3 + 36b^2c^4 - 30a^4bc + 24a^3bc^2 - 36a^2bc^3 + 9a^4c^2$.

Ans. $5a^2b - 3a^2c - 4abc + 6bc^2$.

150. We will conclude this subject with the following remarks: —

(1) A binomial can never be a perfect square, since we know that the square of the most simple polynomial (viz., a binomial) contains three distinct parts, which cannot experience any reduction amongst themselves. Thus, the expression $a^2 + b^2$ is not a perfect square: it wants the term $\pm 2ab$, in order that it should be the square of $a \pm b$.

(2) In order that a trinomial, when arranged, may be a perfect square, its two extreme terms must be squares, and the middle term must be the double product of the square roots of the two others. Therefore, to obtain the square root of a trinomial when it is a perfect square,

Extract the roots of the two extreme terms, and give these roots the same or contrary signs, according as the middle term is positive or negative. To verify it, see if the double product of the two roots is the same as the middle term of the trinomial.

Thus, $9a^6 - 48a^4b^2 + 64a^2b^4$ is a perfect square,

since $\sqrt{9a^6} = 3a^3$ and $\sqrt{64a^2b^4} = -8ab^2$;

and also

$$2 \times 3a^3 \times -8ab^2 = -48a^4b^2 = \text{the middle term.}$$

But $4a^2 + 14ab + 9b^2$ is not a perfect square: for although $4a^2$ and $+9b^2$ are the squares of $2a$ and $3b$, yet $2 \times 2a \times 3b$ is not equal to $14ab$.

(3) In the series of operations required by the general rule, when the first term of one of the remainders is not

exactly divisible by twice the first term of the root, we may conclude that the proposed polynomial is not a perfect square. This is an evident consequence of the course of reasoning by which we have arrived at the general rule for extracting the square root.

(4) When the polynomial is *not a perfect square*, it may sometimes be simplified (see § 139).

Take, for example, the expression $\sqrt{a^3b+4a^2b^2+4ab^3}$.

The quantity under the radical sign is not a perfect square; but it can be put under the form $ab(a^2+4ab+4b^2)$. Now, the factor within the parenthesis is evidently the square of $a+2b$, whence we may conclude that

$$\sqrt{a^3b+4a^2b^2+4ab^3}=(a+2b)\sqrt{ab}.$$

Take also the expression $\sqrt{2a^2b-4ab^2+2b^3}$.

$$\sqrt{2a^2b-4ab^2+2b^3}=(a-b)\sqrt{2b}.$$

CUBE ROOT OF POLYNOMIALS.

150 a. From the rules for forming the powers of binomials, we have

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

Hence the cube of a binomial is arranged with reference to the descending powers of its first term, and is equal to the cube of the first term plus three times the square of the first term multiplied by the second, plus other terms.

In reversing the process, to find the cube root of a polynomial containing not more than four terms, we must arrange it with reference to one of its letters; then find the eube root of the first term for the first term of the root, and subtract its cube from the polynomial. The first term of the remainder will be three times the square of the first term of the root multiplied by the second; and therefore, to *find* the second

term of the root, divide the first term of the remainder by three times the square of the first term of the root.

As an example, extract the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

$$\begin{array}{rl|l} & a^3 + 3a^2b + 3ab^2 + b^3 & \underline{a+b} \\ a^3 = & \underline{a^3 \qquad\qquad\qquad\qquad} & 3a^2, \text{ Divisor.} \\ & \quad 3a^2b + \text{etc.} & \\ (a+b)^3 = & \underline{a^3 + 3a^2b + 3ab^2 + b^3} & \\ & \qquad\qquad 0 \quad \text{The root is exact.} & \end{array}$$

Suppose it be required to raise $x + y + c$ to the third power. We may write it in the *form* of a binomial, thus: $(x + y) + c$, in which the *sum* of x and y is taken for the first term. Then, as before, we have

$$[(x + y) + c]^3 = (x + y)^3 + 3(x + y)^2c + 3(x + y)c^2 + c^3.$$

Now, the cube root of the second member has three terms. The first two terms, x and y, are obtained just as in the preceding example. When the cube of $(x + y)$, or $(x + y)^3$, is subtracted, there will be a second remainder consisting of $3(x + y)^2c$, plus other terms. The *first* term of this remainder is $3x^2c$; that is, it is three times the square of the *first term* of the root multiplied by the *third term* of the root; and hence, to *find* the third term of the root, divide the first term of the second remainder by three times the square of the first term of the root.

To illustrate by an example, extract the cube root of

$$x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1.$$

$$\begin{array}{rl|l} & x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1 & \underline{x^2 - 2x + 1} \\ (x^2 - 2x)^3 = & \underline{x^6 - 6x^5 + 12x^4 - \ \ 8x^3} & 3x^4, \text{Divisor.} \\ & \qquad\qquad 3x^4 - \text{etc.} & \\ (x^2 - 2x + 1)^3 = & \underline{x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1} & \\ & \qquad\qquad\qquad 0 & \end{array}$$

In this example, we first extract the cube root of x^6, which gives x^2, for the first term of the root. Squaring x^2, and multiplying by 3, we obtain the divisor $3x^4$: this is contained in the second term $-2x$ times. Then cubing the sum of these two terms, and subtracting, we find that the first term of the remainder, $3x^4$, contains the divisor once. Cubing the sum of the three terms and subtracting, we find a remainder 0. Hence x^2-2x+1 is the exact cube root.

Hence we have the following rule: —

Arrange the given polynomial with reference to one of its letters, and extract the cube root of the first term. This will be the first term of the root.

Divide the second term of the polynomial by three times the square of the first term of the root. The quotient will be the second term of the root.

Subtract the cube of the sum of the two terms of the root from the given polynomial, and divide the first term of the remainder by three times the square of the first term of the root. The quotient will be the third term of the root.

Continue this operation till a remainder 0 is found, or until one is found whose first term is not divisible by three times the square of the first term of the root.

In the former case the root is exact: in the latter case the polynomial is an imperfect third power.

Exercises.

Find the cube roots of the following polynomials: —

1. $8x^3-12x^2+6x-1$. *Ans.* $2x-1$.
2. $x^3+15x^2+75x+125$. *Ans.* $x+5$.
3. $x^6+6x^5-40x^3+96x-64$. *Ans.* x^2+2x-4.
4. $8x^6-12x^5+30x^4-25x^3+30x^2-12x+8$. *Ans.* $2x^2-x+2$.

5. $64a^6 - 288a^5 + 1080a^3 - 1458a - 729.$

Ans. $4a^2 - 6a - 9.$

6. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6.$

Ans. $1 - 2x + 3x^2.$

By extracting the required root of the first and the last terms, two terms of the root may in general be found, from which the remaining ones may often be determined by inspection. The whole root may then be verified as above.

CHAPTER VIII.

EQUATIONS OF THE SECOND DEGREE.

EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

151. An equation of the second degree containing but one unknown quantity is one in which the greatest exponent of the unknown quantity is equal to 2. Thus,

$$x^2 = a, \quad ax^2 + bx = c,$$

are equations of the second degree.

152. Let us see to what form every equation of the second degree may be reduced.

Take any equation of the second degree, as

$$(1 + x)^2 - \frac{3}{4}x - 10 = 5 - \frac{x}{4} + \frac{x^2}{2}.$$

Clearing of fractions, and performing indicated operations, we have

$$4 + 8x + 4x^2 - 3x - 40 = 20 - x + 2x^2.$$

Transposing the unknown terms to the first member, the known terms to the second, and arranging with reference to the powers of x, we have

$$4x^2 - 2x^2 + 8x - 3x + x = 20 + 40 - 4.$$

By reducing, $\qquad 2x^2 + 6x = 56.$

Dividing by the coefficient of x^2, we have

$$x^2 + 3x = 28.$$

If we denote the coefficient of x by $2p$, and the second member by q, we have

$$x^2 + 2px = q.$$

This is called the **reduced equation.**

153. When the reduced equation is of this form, it contains three terms, and is called a **complete equation.** The terms are, —

FIRST TERM. — The second power of the unknown quantity, with a plus sign.

SECOND TERM. — The first power of the unknown quantity, with a coefficient.

THIRD TERM. — A known term in the second member.

Every equation of the second degree may be reduced to this form by the following rule: —

Clear the equation of fractions, and perform all the indicated operations.

Transpose all the unknown terms to the first member, and all the known terms to the second member.

Reduce all the terms containing the square of the unknown quantity to a single term, one factor of which is the square of the unknown quantity; reduce also all the terms containing the first power of the unknown quantity to a single term.

Divide both members of the resulting equation by the coefficient of the square of the unknown quantity.

154. A **root** of an equation is such a value of the unknown quantity as, being substituted for it, will satisfy the equation; that is, make the two members equal.

The **solution** of an equation is the operation of finding its roots.

INCOMPLETE EQUATIONS.

155. It may happen that $2p$, the coefficient of the first power of x, in the equation $x + 2px = q$, is equal to 0. In this case the first power of x will disappear, and the equation will take the form

$$x^2 = q \tag{1}$$

This is called an **incomplete equation.** Hence

An **incomplete equation,** when reduced, contains but two terms, — the square of the unknown quantity, and a known term.

156. Extracting the square root of both members of Equation (1), we have

$$x = \pm \sqrt{q} \tag{2}$$

Hence, for the solution of incomplete equations,

Reduce the equation to the form $x^2 = q$.

Then extract the square root of both members.

NOTE. — There will be two roots, numerically equal, but having contrary signs. Denoting the first by x', and the second by x'', we have

$$x' = +\sqrt{q}, \text{ and } x'' = -\sqrt{q}.$$

VERIFICATION. — Substituting $+\sqrt{q}$ or $-\sqrt{q}$ for x, in Equation (1), we have $(+\sqrt{q})^2 = q$, and $(-\sqrt{q})^2 = q$. Hence both satisfy the equation: they are therefore roots (§ 154).

Exercises.

1. What are the values of x in the equation $3x^2 + 8 = 5x^2 - 10$?

By transposing,	$3x^2 - 5x^2 = -10 - 8.$
Reducing,	$-2x^2 = -18.$
Dividing by -2,	$x^2 = 9.$
Extracting square root,	$x = \pm\sqrt{9} = +3$ and $-3.$
Hence	$x' = +3$, and $x'' = -3.$

2. What are the roots of the equation $3x^2 + 6 = 4x^2 - 10$?
Ans. $x' = +4,\ x'' = -4.$

3. What are the roots of the equation $\frac{1}{3}x^2 - 8 = \frac{x^2}{9} + 10$?
Ans. $x' = +9,\ x'' = -9.$

4. What are the roots of the equation $4x^2 + 13 - 2x^2 = 45$?
Ans. $x' = +4,\ x'' = -4.$

5. What are the roots of the equation $6x^2 - 7 = 3x^2 + 5$?
Ans. $x' = +2,\ x'' = -2.$

6. What are the roots of the equation $8 + 5x^2 = \frac{x^2}{5} + 4x^2 + 28$?
Ans. $x' = +5,\ x'' = -5.$

7. What are the roots of the equation $\frac{3x^2+5}{8} - \frac{x^2+29}{3} = 117 - 5x^2$?
Ans. $x' = +5,\ x'' = -5.$

8. What are the roots of the eqùation $x^2 + ab = 5x^2$?
Ans. $x' = +\frac{1}{2}\sqrt{ab},\ x'' = -\frac{1}{2}\sqrt{ab}.$

9. What are the roots of the equation $x\sqrt{a + x^2} = b + x^2$?
Ans. $x' = \frac{b}{\sqrt{a - 2b}},\ x'' = -\frac{b}{\sqrt{a - 2b}}.$

Problems for Solution.

1. What number is that which being multiplied by itself the product will be 144?

Let $x =$ the number.

Then $x \times x = x^2 = 144.$

It is plain that the value of x will be found by extracting the square root of both members of the equation; that is,

$$\sqrt{x^2} = \sqrt{144};\ \text{ that is, } x = 12.$$

2. A person, being asked how much money he had, said, "If the number of dollars be squared and 6 be added, the sum will be 42." How much had he? *Ans.* \$6.

Let $x =$ the number of dollars.

Then, by the conditions, $x^2 + 6 = 42.$

Hence $x^2 = 42 - 6 = 36,$

and $x = 6.$

3. A grocer, being asked how much sugar he had sold to a person, answered, "If the square of the number of pounds be multiplied by 7, the product will be 1575." How many pounds had he sold? *Ans.* 15.

Let $x =$ the number of pounds.

Then, by the conditions of the question,

$$7x^2 = 1575.$$

Hence $x^2 = 225,$

and $x = 15.$

4. A person, being asked his age, said, "If from the square of my age in years you take 192 years, the remainder will be the square of half my age." What was his age?

Ans. 16 years.

Let $x =$ the number of years in his age.

Then, by the conditions of the question,

$$x^2 - 192 = \left(\frac{1}{2}x\right)^2 = \frac{x^2}{4}.$$

Clearing of fractions, $4x^2 - 768 = x^2.$

Hence $4x^2 - x^2 = 768,$

and $3x^2 = 768.$

$$x^2 = 256.$$

$$x = 16.$$

5. What number is that whose eighth part multiplied by its fifth part, and the product divided by 4, will give a quotient equal to 40? *Ans.* 80.

Let $x =$ the number.

By the conditions of the question,

$$\left(\frac{1}{8}x \times \frac{1}{5}x\right) \div 4 = 40.$$

Hence $$\frac{x^2}{160} = 40.$$

Clearing of fractions, $$x^2 = 6400.$$

$$x = 80.$$

6. Find a number such that one third of it multiplied by one fourth shall be equal to 108. *Ans.* 36.

7. What number is that whose sixth part multiplied by its fifth part, and the product divided by 10, will give a quotient equal to 3? *Ans.* 30.

8. What number is that whose square plus 18 will be equal to half the square plus $30\frac{1}{2}$? *Ans.* 5.

9. What numbers are those which are to each other as 1 to 2, and the difference of whose squares is equal to 75?

Ans. 5 and 10.

Let $x =$ the less number.

Then $2x =$ the greater.

Then, by the conditions of the question,

$$4x^2 - x^2 = 75.$$

Hence $$3x^2 = 75.$$

Dividing by 3, $$x^2 = 25, \text{ and } x = 5,$$

and $$2x = 10.$$

10. What two numbers are those which are to each other as 5 to 6, and the difference of whose squares is 44?

Ans. 10 and 12.

Let $x =$ the greater number.

Then $\frac{5}{6}x =$ the less.

By the conditions of the problem,

$$x^2 - \frac{25}{36}x^2 = 44.$$

Clearing of fractions, $36x^2 - 25x^2 = 1584.$

Hence $11x^2 = 1584,$

and $x^2 = 144.$

Hence $x = 12,$

and $\frac{5}{6}x = 10.$

11. What two numbers are those which are to each other as 3 to 4, and the difference of whose squares is 28?

Ans. 6 and 8.

12. What two numbers are those which are to each other as 5 to 11, and the sum of whose squares is 584?

Ans. 10 and 22.

13. A says to B, "My son's age is one quarter of yours, and the difference between the squares of the numbers representing their ages is 240." What were their ages?

Ans. Elder, 16; younger, 4.

Equations containing Two Unknown Quantities.

157. When there are two or more unknown quantities,

Eliminate one of the unknown quantities by § 113.

Then extract the square root of both members of the equation.

PROBLEMS FOR SOLUTION.

1. There is a room of such dimensions that the difference of the sides multiplied by the less is equal to 36, and the product of the sides is equal to 360. What are the sides?

Ans. $x = 18$, $y = 20$.

Let $x =$ the length of the less side,

and $y =$ the length of the greater.

Then, by the 1st condition, $(y - x)x = 36$;

and by the 2d, $xy = 360$.

From the first equation we have

$$xy - x^2 = 36;$$

and by subtraction, $x^2 = 324$.

Hence $x = \sqrt{324} = 18$,

and $y = \dfrac{360}{18} = 20$.

2. A merchant sells two pieces of muslin, which together measure 12 yards. He received for each piece just as many dollars per yard as the piece contained yards. Now, he gets four times as much for one piece as for the other. How many yards in each piece?

Ans. 8 and 4.

Let $x =$ the number of yards in the longer piece,

and $y =$ the number of yards in the shorter piece.

Then, by the conditions of the question,

$$x + y = 12.$$

$x \times x = x^2 =$ what he got for the longer piece.

$y \times y = y^2 =$ what he got for the shorter.

$x^2 = 4y^2$, by the 2d condition.

$x = 2y$, by extracting the square root.

Substituting this value of x in the first equation, we have

$$y + 2y = 12,$$

and consequently $y = 4$,

and $x = 8$.

3. What two numbers are those whose product is 30, and the quotient of the greater by the less, $3\frac{1}{3}$? *Ans.* 10 and 3.

4. The product of two numbers is a, and their quotient b. What are the numbers?

Ans. $\sqrt{ab}$ and $\sqrt{\frac{a}{b}}$.

5. The sum of the squares of two numbers is 117, and the difference of their squares 45. What are the numbers?

Ans. 9 and 6.

6. The sum of the squares of two numbers is a, and the difference of their squares is b. What are the numbers?

Ans. $\sqrt{\frac{a+b}{2}}$ and $\sqrt{\frac{a-b}{2}}$.

7. What two numbers are those which are to each other as 3 to 4, and the sum of whose squares is 225?

Ans. 9 and 12.

8. What two numbers are those which are to each other as m to n, and the sum of whose squares is equal to a^2?

Ans. $\frac{ma}{\sqrt{m^2+n^2}}$ and $\frac{na}{\sqrt{m^2+n^2}}$.

9. What two numbers are those which are to each other as 1 to 2, and the difference of whose squares is 75?

Ans. 5 and 10.

10. What two numbers are those which are to each other as m to n, and the difference of whose squares is equal to b^2?

Ans. $\frac{mb}{\sqrt{m^2-n^2}}$ and $\frac{nb}{\sqrt{m^2-n^2}}$.

11. A certain sum of money is placed at interest for six months, at 8 per cent per annum. Now, if the sum put at interest be multiplied by the number expressing the interest, the product will be \$562500. What is the principal at interest? *Ans.* \$3750.

12. A person distributes a sum of money between a number of women and boys. The number of women is to the number of boys as 3 to 4. Now, the boys receive one half as many dollars as there are persons; and the women, twice as many dollars as there are boys; and together they receive $188. How many women were there, and how many boys?

Ans. 36 women, 48 boys.

Complete Equations.

158. The reduced form of the complete equation (§ 153) is

$$x^2 + 2px = q.$$

Comparing the first member of this equation with the square of a binomial (§ 54), we see that it needs but the square of half the coefficient of x to make it a perfect square. Adding p^2 to both members (§ 102, Axiom 1), we have

$$x^2 + 2px + p^2 = q + p^2.$$

Then, extracting the square root of both members (Axiom 5), we have

$$x + p = \pm\sqrt{q + p^2}.$$

Transposing p to the second member, we have

$$x = -p \pm\sqrt{q + p^2}.$$

Hence there are two roots, — one corresponding to the *plus* sign of the radical, and the other to the *minus* sign. Denoting these roots by x' and x'', we have

$$x' = -p + \sqrt{q + p^2}, \text{ and } x'' = -p - \sqrt{q + p^2}.$$

The root denoted by x' is called the **first root**; that denoted by x'' is called the **second root**.

159. Completing the square is the operation of squaring half the coefficient of x, and adding the result to both members of the equation. For the solution of every complete equation of the second degree, we have the following rule: —

Reduce the equation to the form $x^2 + 2px = q$.

Take half the coefficient of the second term, square it, and add the result to both members of the equation.

Then extract the square root of both members, after which transpose the known term to the second member.

NOTE. — Although in the beginning the student should complete the square and then extract the square root, yet he should be able in all cases to write the roots immediately by the following (see § 158) rule: —

The first root is equal to half the coefficient of the second term of the reduced equation, taken with a contrary sign, plus the square root of the second member increased by the square of half the coefficient of the second term.

The second root is equal to half the coefficient of the second term of the reduced equation, taken with a contrary sign, minus the square root of the second member increased by the square of half the coefficient of the second term.

160. We will now show that the complete equation of the second degree will take four forms, dependent on the signs of $2p$ and q.

Let us suppose $2p$ to be positive, and q positive: we shall then have

$$x^2 + 2px = q \tag{1}$$

Let us suppose $2p$ to be negative, and q positive: we shall then have

$$x^2 - 2px = q \tag{2}$$

Let us suppose $2p$ to be positive, and q negative: we shall then have

$$x^2 + 2px = -q \tag{3}$$

Let us suppose $2p$ to be negative, and q negative: we shall then have

$$x^2 - 2px = -q \qquad (4)$$

As these are all the combinations of signs that can take place between $2p$ and q, we conclude that every complete equation of the second degree will be reduced to one or the other of these four forms: —

$$x^2 + 2px = +q, \qquad \text{1st form.}$$
$$x^2 - 2px = +q, \qquad \text{2d form.}$$
$$x^2 + 2px = -q, \qquad \text{3d form.}$$
$$x^2 - 2px = -q, \qquad \text{4th form.}$$

Exercises.

First Form.

1. What are the values of x in the equation $2x^2 + 8x = 64$?

If we first divide by the coefficient 2, we obtain

$$x^2 + 4x = 32.$$

Completing the square, $x^2 + 4x + 4 = 32 + 4 = 36.$

Extracting the root, $x + 2 = \pm\sqrt{36} = +6$ and -6.

Hence $x' = -2 + 6 = +4,$

and $x'' = -2 - 6 = -8.$

Hence, in this form, the smaller root, numerically, is positive, and the larger negative.

Verification. If we take the positive value (viz., $x' = +4$), the equation $x^2 + 4x = 32$

gives $4^2 + 4 \times 4 = 32;$

and if we take the negative value of x (viz., $x'' = -8$), the equation

$$x^2 + 4x = 32$$

gives $(-8)^2 + 4(-8) = 64 - 32 = 32;$

from which we see that either of the values of x (viz.,

$$x' = +4, \text{ or } x'' = -8)$$

will satisfy the equation.

2. What are the values of x in the equation

$$3x^2+12x-19=-x^2-12x+89?$$

Transposing the terms, we have

$$3x^2+x^2+12x+12x=89+19.$$

Reducing, $$4x^2+24x=108.$$

Dividing by the coefficient of x^2,

$$x^2+6x=27.$$

Completing the square, $$x^2+6x+9=36.$$

Extracting the square root,

$$x+3=\pm\sqrt{36}=+6 \text{ and } -6.$$

Hence $$x'=+6-3=+3,$$

and $$x''=-6-3=-9.$$

VERIFICATION. If we take the plus root, the equation

$$x^2+6x=27$$

gives $$(3)^2+6(3)=27.$$

If we take the negative root, $x^2+6x=27$

gives $$(-9)^2+6(-9)=81-54=27.$$

3. What are the values of x in the equation

$$x^2-10x+15=\frac{x^2}{5}-34x+155?$$

Clearing of fractions, we have

$$5x^2-50x+75=x^2-170x+775.$$

Transposing and reducing, we obtain,

$$4x^2+120x=700.$$

Dividing by the coefficient of x^2, we have

$$x^2+30x=175.$$

Completing the square, $x^2+30x+225=400.$

Extracting the square root,

$$x + 15 = \pm\sqrt{400} = +20 \text{ and } -20.$$

Hence $\qquad x' = +5, \text{ and } x'' = -35.$

VERIFICATION. For the plus value of x, the equation

$$x^2 + 30x = 175$$

gives $\qquad (5)^2 + 30 \times 5 = 25 + 150 = 175.$

For the negative value of x, we have

$$(-35)^2 + 30(-35) = 1225 - 1050 = 175.$$

4. What are the values of x in the equation

$$\frac{5}{6}x^2 - \frac{1}{2}x + \frac{3}{4} = 8 - \frac{2}{3}x - x^2 + \frac{273}{12}?$$

Clearing of fractions, we have

$$10x^2 - 6x + 9 = 96 - 8x - 12x^2 + 273.$$

Transposing and reducing, $22x^2 + 2x = 360.$

Dividing both members by 22,

$$x^2 + \frac{2}{22}x = \frac{360}{22}.$$

Add $\left(\frac{1}{22}\right)^2$ to both members, and the equation becomes

$$x^2 + \frac{2}{22}x + \left(\frac{1}{22}\right)^2 = \frac{360}{22} + \left(\frac{1}{22}\right)^2.$$

Whence, by extracting the square root,

$$x + \frac{1}{22} = \pm\sqrt{\frac{360}{22} + \left(\frac{1}{22}\right)^2}.$$

$$\therefore\ x' = -\frac{1}{22} + \sqrt{\frac{360}{22} + \left(\frac{1}{22}\right)^2},$$

and $\qquad x'' = -\frac{1}{22} - \sqrt{\frac{360}{22} + \left(\frac{1}{22}\right)^2}.$

It remains to perform the numerical operations. In the first place,

$$\frac{360}{22} + \left(\frac{1}{22}\right)^2$$

must be reduced to a single number, having $(22)^2$ for its denominator.

Now, $\qquad \frac{360}{22} + \left(\frac{1}{22}\right)^2 = \frac{360 \times 22 + 1}{(22)^2} = \frac{7921}{(22)^2}.$

Extracting the square root of 7921, we find it to be 89.

$$\therefore \pm\sqrt{\frac{360}{22}+\left(\frac{1}{22}\right)^2}=\pm\frac{89}{22}.$$

Consequently the plus value of x is

$$x'=-\frac{1}{22}+\frac{89}{22}=\frac{88}{22}=4,$$

and the negative value is

$$x''=-\frac{1}{22}-\frac{89}{22}=-\frac{45}{11};$$

that is, one of the two values of x which will satisfy the proposed equation is a positive whole number, and the other a negative fraction.

NOTE. — Let the pupil be exercised in writing the roots in the last five and in the following examples *without completing the square.*

5. What are the values of x in the equation

$$3x^2+2x-9=76? \qquad Ans. \begin{cases} x'=5. \\ x''=-5\frac{2}{3}. \end{cases}$$

6. What are the values of x in the equation

$$2x^2+8x+7=\frac{5x}{4}-\frac{x^2}{8}+197? \qquad Ans. \begin{cases} x'=8. \\ x''=-11\frac{3}{17}. \end{cases}$$

7. What are the values of x in the equation

$$\frac{x^2}{4}-\frac{x}{3}+15=\frac{x^2}{9}-8x+95\frac{1}{4}? \qquad Ans. \begin{cases} x'=9. \\ x''=-64\frac{1}{5}. \end{cases}$$

8. What are the values of x in the equation

$$\frac{x^2}{1}-\frac{5x}{4}-8=\frac{x}{2}-7x+6\frac{1}{2}? \qquad Ans. \begin{cases} x'=2. \\ x''=-7\frac{1}{4}. \end{cases}$$

9. What are the values of x in the equation

$$\frac{x^2}{2}+\frac{x}{4}=\frac{x^2}{5}-\frac{x}{10}+\frac{13}{20}? \qquad Ans. \begin{cases} x'=1. \\ x''=-2\frac{1}{6}. \end{cases}$$

SECOND FORM.

1. What are the values of x in the equation

$$x^2 - 8x + 10 = 19?$$

Transposing, $\quad x^2 - 8x = 19 - 10 = 9.$

Completing the square,

$$x^2 - 8x + 16 = 9 + 16 = 25.$$

Extracting the root,

$$x - 4 = \pm\sqrt{25} = +5 \text{ or } -5.$$

Hence $\quad x' = 4 + 5 = 9,$

and $\quad x'' = 4 - 5 = -1.$

That is, in this form, the larger root, numerically, is positive, and the lesser negative.

VERIFICATION. If we take the positive value of x, the equation

$$x^2 - 8x = 9$$

gives $\quad (9)^2 - 8 \times 9 = 81 - 72 = 9.$

If we take the negative value, the equation

$$x^2 - 8x = 9$$

gives $\quad (-1)^2 - 8(-1) = 1 + 8 = 9,$

from which we see that both roots alike satisfy the equation.

2. What are the values of x in the equation

$$\frac{x^2}{2} + \frac{x}{3} - 15 = \frac{x^2}{4} + x - 14\tfrac{3}{4}?$$

Clearing of fractions, we have

$$6x^2 + 4x - 180 = 3x^2 + 12x - 177.$$

Transposing and reducing,

$$3x^2 - 8x = 3.$$

Dividing by the coefficient of x^2, we obtain

$$x^2 - \frac{8}{3}x = 1.$$

Completing the square, we have

$$x^2 - \frac{8}{3}x + \frac{16}{9} = 1 + \frac{16}{9} = \frac{25}{9}.$$

Extracting the square root,

$$x - \frac{4}{3} = \pm\sqrt{\frac{25}{9}} = +\frac{5}{3} \text{ and } -\frac{5}{3}.$$

Hence $\quad x' = \frac{4}{3} + \frac{5}{3} = +3$, and $x'' = \frac{4}{3} - \frac{5}{3} = -\frac{1}{3}.$

VERIFICATION. For the positive root of x, the equation

$$x^2 - \frac{8}{3}x = 1$$

gives

$$3^2 - \frac{8}{3} \times 3 = 9 - 8 = 1.$$

For the negative root, the equation

$$x^2 - \frac{8}{3}x = 1$$

gives

$$\left(-\frac{1}{3}\right)^2 - \frac{8}{3} \times -\frac{1}{3} = \frac{1}{9} + \frac{8}{9} = 1.$$

3. What are the values of x in the equation

$$\frac{x^2}{2} - \frac{x}{3} + 7\tfrac{3}{8} = 8\,?$$

Clearing of fractions, and dividing by the coefficient of x, we have

$$x^2 - \frac{2}{3}x = 1\tfrac{1}{4}.$$

Completing the square, we have

$$x^2 - \frac{2}{3}x + \frac{1}{9} = 1\tfrac{1}{4} + \frac{1}{9} = \frac{49}{36}.$$

Extracting the square root, we have

$$x - \frac{1}{3} = \pm\sqrt{\frac{49}{36}} = +\frac{7}{6} \text{ and } -\frac{7}{6}.$$

Hence $\quad x' = \frac{1}{3} + \frac{7}{6} = \frac{9}{6} = 1\tfrac{1}{2}$, and $x'' = \frac{1}{3} - \frac{7}{6} = -\frac{5}{6}.$

VERIFICATION. If we take the positive root of x, the equation

$$x^2 - \frac{2}{3}x = 1\tfrac{1}{4}$$

gives $$(1\tfrac{1}{2})^2 - \frac{2}{3} \times 1\tfrac{1}{2} = 2\tfrac{1}{4} - 1 = 1\tfrac{1}{4}.$$

If we take the negative root, the equation

$$x^2 - \frac{2}{3}x = 1\tfrac{1}{4}$$

gives $$\left(-\frac{5}{6}\right)^2 - \frac{2}{3} \times -\frac{5}{6} = \frac{25}{36} + \frac{10}{18} = \frac{45}{36} = 1\tfrac{1}{4}.$$

4. What are the values of x in the equation

$$4a^2 - 2x^2 + 2ax = 18ab - 18b^2?$$

By transposing, changing the signs, and dividing by 2, the equation becomes $$x^2 - ax = 2a^2 - 9ab + 9b^2.$$

Completing the square, $$x^2 - ax + \frac{a^2}{4} = \frac{9a^2}{4} - 9ab + 9b^2.$$

Extracting the square root, $$x = \frac{a}{2} \pm \sqrt{\frac{9a^2}{4} - 9ab + 9b^2}.$$

Now, the square root of $\frac{9a^2}{4} - 9ab + 9b^2$ is evidently $\frac{3a}{2} - 3b$.

$$\therefore\ x = \frac{a}{2} \pm \left(\frac{3a}{2} - 3b\right), \text{ and } \begin{cases} x' = 2a - 3b. \\ x'' = -a + 3b. \end{cases}$$

What will be the numerical values of x, if we suppose $a = 6$, and $b = 1$?

5. What are the values of x in the equation

$$\frac{1}{3}x - 4 - x^2 + 2x - \frac{4}{5}x^2 = 45 - 3x^2 + 4x?$$

Ans. $\left\{\begin{array}{l} x' = 7.12 \\ x'' = -5.73 \end{array}\right\}$ to within 0.01.

6. What are the values of x in the equation

$$8x^2 - 14x + 10 = 2x + 34?$$ *Ans.* $\begin{cases} x' = 3. \\ x'' = -1. \end{cases}$

7. What are the values of x in the equation

$$\frac{x^2}{4} - 30 + x = 2x - 22? \qquad Ans. \begin{cases} x' = 8. \\ x'' = -4. \end{cases}$$

8. What are the values of x in the equation

$$x^2 - 3x + \frac{x^2}{2} = 9x + 13\tfrac{1}{2}? \qquad Ans. \begin{cases} x' = 9. \\ x'' = -1. \end{cases}$$

9. What are the values of x in the equation

$$2ax - x^2 = -2ab - b^2? \qquad Ans. \begin{cases} x' = 2a + b. \\ x'' = -b. \end{cases}$$

10. What are the values of x in the equation

$$a^2 + b^2 - 2bx + x^2 = \frac{m^2x^2}{n^2}?$$

$$Ans. \begin{cases} x' = \dfrac{n}{n^2 - m^2}(bn + \sqrt{a^2m^2 + b^2m^2 - a^2n^2}). \\[2ex] x'' = \dfrac{n}{n^2 - m^2}(bn - \sqrt{a^2m^2 + b^2m^2 - a^2n^2}). \end{cases}$$

Third Form.

1. What are the values of x in the equation $x^2 + 4x = -3$?

Completing the square, we have

$$x^2 + 4x + 4 = -3 + 4 = 1.$$

Extracting the square root,

$$x + 2 = \pm\sqrt{1} = +1, \text{ and } -1.$$

Hence $$x' = -2 + 1 = -1;$$

and $$x'' = -2 - 1 = -3;$$

that is, in this form both the roots are negative.

Verification. If we take the first negative value, the equation

$$x^2 + 4x = -3$$

gives $$(-1)^2 + 4(-1) = 1 - 4 = -3.$$

If we take the second value, the equation

$$x^2 + 4x = -3$$

gives $\qquad (-3)^2 + 4(-3) = 9 - 12 = -3.$

Hence both values of x satisfy the given equation.

2. What are the values of x in the equation

$$-\frac{x^2}{2} - 5x - 16 = 12 + \frac{1}{2}x^2 + 6x\,?$$

Transposing and reducing, we have

$$-x^2 - 11x = 28.$$

Dividing by -1, the coefficient of x^2, we have

$$x^2 + 11x = -28.$$

Completing the square,

$$x^2 + 11x + 30.25 = 2.25.$$

Hence $\qquad x + 5.5 = \pm\sqrt{2.25} = +1.5$ and $-1.5.$

Consequently $\qquad x' = -4$, and $x'' = -7.$

3. What are the values of x in the equation

$$-\frac{x^2}{8} - 2x - 5 = \frac{7}{8}x^2 + 5x + 5\,? \qquad \textit{Ans.} \begin{cases} x' = -2. \\ x'' = -5. \end{cases}$$

4. What are the values of x in the equation

$$2x^2 + 8x = -2\tfrac{2}{3} - \frac{2}{3}x\,? \qquad \textit{Ans.} \begin{cases} x' = -\frac{1}{3}. \\ x'' = -4. \end{cases}$$

5. What are the values of x in the equation

$$4x^2 + \frac{3}{5}x + 3x = -14x - 3\tfrac{1}{5} - 4x^2\,? \qquad \textit{Ans.} \begin{cases} x' = -\frac{1}{5}. \\ x'' = -2. \end{cases}$$

6. What are the values of x in the equation

$$-x^2 - 4 - \frac{3}{4}x = \frac{4x^2}{2} + 24x + 2\,? \qquad \textit{Ans.} \begin{cases} x' = -\frac{1}{4}. \\ x'' = -8. \end{cases}$$

7. What are the values of x in the equation

$\frac{1}{9}x^2+7x+20=-\frac{8}{9}x^2-11x-60$? *Ans.* $\begin{cases} x' = -\ 8. \\ x'' = -10. \end{cases}$

8. What are the values of x in the equation

$\frac{5}{6}x^2-x+\frac{1}{2}=-9\frac{1}{8}x-\frac{1}{6}x^2-\frac{1}{2}$? *Ans.* $\begin{cases} x' = -\frac{1}{8}. \\ x'' = -8. \end{cases}$

9. What are the values of x in the equation

$\frac{4}{5}x^2+5x+\frac{1}{4}=-\frac{1}{5}x^2-5\frac{1}{10}x-\frac{3}{4}$? *Ans.* $\begin{cases} x' = -\frac{1}{10}. \\ x'' = -10. \end{cases}$

10. What are the values of x in the equation

$x-x^2-3=6x+1$? *Ans.* $\begin{cases} x' = -1. \\ x'' = -4. \end{cases}$

11. What are the values of x in the equation

$x^2+4x-90=-93$? *Ans.* $\begin{cases} x' = -1. \\ x'' = -3. \end{cases}$

Fourth Form.

1. What are the values of x in the equation $x^2-8x=-7$?

Completing the square, we have

$$x^2-8x+16=-7+16=9.$$

Extracting the square root,

$$x-4=\pm\sqrt{9}=+3 \text{ and } -3.$$

Hence $\qquad x'=+7$, and $x''=+1$;

that is, in this form both the roots are positive.

Verification. If we take the greater root, the equation

$$x^2-8x=-7$$

gives $\qquad 7^2-8\times 7=49-56=-7.$

If we take the lesser, the equation

$$x^2-8x=-7$$

gives $\qquad 1^2-8\times 1=\ 1-\ 8=-7.$

Hence both of the roots will satisfy the equation.

2. What are the values of x in the equation

$$-1\tfrac{1}{2}x^2+3x-10=1\tfrac{1}{2}x^2-18x+\frac{40}{2}?$$

Clearing of fractions, we have

$$-3x^2+6x-20=3x^2-36x+40.$$

Collecting the similar terms,

$$-6x^2+42x=60.$$

Dividing by the coefficient of x^2, which is -6, we have

$$x^2-7x=-10.$$

Completing the square, we have

$$x^2-7x+12.25=2.25.$$

Extracting the square root of both members,

$$x-3.5=\pm\sqrt{2.25}=+1.5 \text{ and } -1.5.$$

Hence $\quad x'=3.5+1.5=5$, and $x''=3.5-1.5=2$.

VERIFICATION. If we take the greater root, the equation

$$x^2-7x=-10$$

gives $\quad 5^2-7\times 5=25-35=-10.$

If we take the lesser root, the equation

$$x^2-7x=-10$$

gives $\quad 2^2-7\times 2=4-14=-10.$

3. What are the values of x in the equation

$$-3x+2x^2+1=17\tfrac{4}{5}x-2x^2-3?$$

Transposing and collecting the terms, we have

$$4x^2-20\tfrac{4}{5}x=-4.$$

Dividing by the coefficient of x^2, we have

$$x^2-5\tfrac{1}{5}x=-1.$$

Completing the square, we obtain

$$x^2-5\tfrac{1}{5}x+\frac{169}{25}=-1+\frac{169}{25}=\frac{144}{25}.$$

Extracting the root,

$$x^2 - 2\tfrac{3}{5} = \pm\sqrt{\frac{144}{25}} = +\frac{12}{5} \text{ and } -\frac{12}{5}.$$

Hence $\quad x' = 2\tfrac{3}{5} + \dfrac{12}{5} = 5$, and $x'' = 2\tfrac{3}{5} - \dfrac{12}{5} = \dfrac{1}{5}.$

VERIFICATION. If we take the greater root, the equation

$$x^2 - 5\tfrac{1}{5}x = -1$$

gives $$5^2 - 5\tfrac{1}{5} \times 5 = 25 - 26 = -1.$$

If we take the lesser root, the equation

$$x^2 - 5\tfrac{1}{5}x = -1$$

gives $$\left(\frac{1}{5}\right)^2 - 5\tfrac{1}{5} \times \frac{1}{5} = \frac{1}{25} - \frac{26}{25} = -1.$$

4. What are the values of x in the equation

$$\frac{1}{7}x^2 - 3x + \frac{1}{2} = -\frac{6}{7}x^2 + \frac{1}{4}x - \frac{1}{4}? \qquad Ans. \begin{cases} x' = 3. \\ x'' = \dfrac{1}{4}. \end{cases}$$

5. What are the values of x in the equation

$$-4x^2 - \frac{1}{7}x + 1\tfrac{1}{7} = -5x^2 + 8x? \qquad Ans. \begin{cases} x' = 8. \\ x'' = \dfrac{1}{7}. \end{cases}$$

6. What are the values of x in the equation

$$-4x^2 + \frac{8}{20}x - \frac{1}{40} = -3x^2 - \frac{1}{20}x + \frac{1}{40}? \qquad Ans. \begin{cases} x' = \dfrac{1}{4}. \\ x'' = \dfrac{1}{5}. \end{cases}$$

7. What are the values of x in the equation

$$x^2 - 10\tfrac{1}{10}x = -1? \qquad Ans. \begin{cases} x' = 10. \\ x'' = \dfrac{1}{10}. \end{cases}$$

8. What are the values of x in the equation

$$-27x + \frac{17x^2}{5} + 100 = \frac{2x^2}{5} + 12x - 26? \quad Ans. \begin{cases} x' = 7. \\ x'' = 6. \end{cases}$$

9. What are the values of x in the equation

$$\frac{8x^2}{3} - 22x + 15 = -\frac{7x^2}{3} + 28x - 30? \quad \textit{Ans.} \begin{cases} x' = 9. \\ x'' = 1. \end{cases}$$

10. What are the values of x in the equation

$$2x^2 - 30x + 3 = -x^2 + 3\tfrac{8}{10}x - \frac{3}{10}? \quad \textit{Ans.} \begin{cases} x' = 11. \\ x'' = \frac{1}{10}. \end{cases}$$

Properties of Equations of the Second Degree.

First Property.

161. We have seen (§ 153) that every complete equation of the second degree may be reduced to the form

$$x^2 + 2px = q \tag{1}$$

Completing the square, we have

$$x^2 + 2px + p^2 = q + p^2.$$

Transposing $q + p^2$ to the first member,

$$x^2 + 2px + p^2 - (q + p^2) = 0 \tag{2}$$

Now, since $x^2 + 2px + p^2$ is the square of $x + p$, and $q + p^2$ the square of $\sqrt{q + p^2}$, we may regard the first member as the difference between two squares. Factoring (§ 56), we have

$$(x + p + \sqrt{q + p^2})(x + p - \sqrt{q + p^2}) = 0 \tag{3}$$

This equation can be satisfied only in two ways,—first, by attributing such a value to x as shall render the first factor equal to 0; or, second, by attributing such a value to x as shall render the second factor equal to 0.

Placing the second factor equal to 0, we have

$$x + p - \sqrt{q + p^2} = 0; \text{ and } x' = -p + \sqrt{q + p^2} \tag{4}$$

Placing the first factor equal to 0, we have

$$x+p+\sqrt{q+p^2}=0;\ \text{and}\ x''=-p-\sqrt{q+p^2} \qquad (5)$$

Since every supposition that will satisfy Equation (3) will also satisfy Equation (1), from which it was derived, it follows that x' and x'' are roots of Equation (1); also that

Every equation of the second degree has two roots, and only two.

NOTE.— The two roots denoted by x' and x'' are the same as found in § 158.

Second Property.

162. We have seen (§ 161) that every equation of the second degree may be placed under the form

$$(x+p+\sqrt{q+p^2})(x+p-\sqrt{q+p^2})=0.$$

By examining this equation, we see that the first factor may be obtained by subtracting the *second* root from the unknown quantity x; and the second factor, by subtracting the *first* root from the unknown quantity x. Hence

Every equation of the second degree may be resolved into two binomial factors of the first degree; the first terms in both factors being the unknown quantity, and the second terms, the roots of the equation taken with contrary signs.

Third Property.

163. If we add Equations (4) and (5), § 161, we have

$$\begin{array}{l} x' = -\ p+\sqrt{q+p^2} \\ x'' = -\ p-\sqrt{q+p^2} \\ \hline x'+x'' = -2p; \end{array} \qquad \text{that is,}$$

In every reduced equation of the second degree the sum of the two roots is equal to the coefficient of the second term taken with a contrary sign.

Fourth Property.

164. If we multiply Equations (4) and (5), § 161, member by member, we have

$$x' \times x'' = (-p + \sqrt{q + p^2})(-p - \sqrt{q + p^2})$$
$$= p^2 - (q + p^2) = -q; \text{ that is,}$$

In every reduced equation of the second degree the product of the two roots is equal to the second member taken with a contrary sign.

Formation of Equations of the Second Degree.

165. By taking the converse of the second property (§ 162), we can form equations which shall have given roots; that is, if they are known, we can find the corresponding equations by the following rule: —

Subtract each root from the unknown quantity.

Multiply the results together, and place their product equal to 0.

Exercises.

NOTE. — Let the pupil prove, *in every case*, that the roots will satisfy the third and fourth properties.

1. If the roots of an equation are 4 and -5, what is the equation? *Ans.* $x^2 + x = 20$.

2. What is the equation when the roots are 1 and -3? *Ans.* $x^2 + 2x = 3$.

3. What is the equation when the roots are 9 and -10? *Ans.* $x^2 + x = 90$.

4. What is the equation whose roots are 6 and -10?
Ans. $x^2 + 4x = 60$.

5. What is the equation whose roots are 4 and -3?
Ans. $x^2 - x = 12$.

6. What is the equation whose roots are 10 and $-\frac{1}{10}$?
Ans. $x^2 - 9\frac{9}{10}x = 1$.

7. What is the equation whose roots are 8 and -2?
Ans. $x^2 - 6x = 16$.

8. What is the equation whose roots are 16 and -5?
Ans. $x^2 - 11x = 80$.

9. What is the equation whose roots are -4 and -5?
Ans. $x^2 + 9x = -20$.

10. What is the equation whose roots are -6 and -7?
Ans. $x^2 + 13x = -42$.

11. What is the equation whose roots are $-\frac{3}{4}$ and -2?
Ans. $x^2 + 2\frac{3}{4}x = -\frac{3}{2}$.

12. What is the equation whose roots are -2 and -3?
Ans. $x^2 + 5x = -6$.

13. What is the equation whose roots are 4 and 3?
Ans. $x^2 - 7x = -12$.

14. What is the equation whose roots are 12 and 2?
Ans. $x^2 - 14x = -24$.

15. What is the equation whose roots are 18 and 2?
Ans. $x^2 - 20x = -36$.

16. What is the equation whose roots are 14 and 3?
Ans. $x^2 - 17x = -42$.

17. What is the equation whose roots are $\frac{4}{9}$ and $-\frac{9}{4}$?

Ans. $x^2 + \frac{65}{36}x = 1.$

18. What is the equation whose roots are 5 and $-\frac{2}{3}$?

Ans. $x^2 - \frac{13}{3}x = \frac{10}{3}.$

19. What is the equation whose roots are a and b?

Ans. $x^2 - (a + b)x = -ab.$

20. What is the equation whose roots are c and $-d$?

Ans. $x^2 - (c - d)x = cd.$

TRINOMIAL EQUATIONS OF THE SECOND DEGREE.

165 a. A trinomial equation of the second degree contains three kinds of terms: —

FIRST TERM. — A term involving the unknown quantity to the second degree.

SECOND TERM. — A term involving the unknown quantity to the first degree.

THIRD TERM. — A known term.

Thus, $$x^2 - 4x - 12 = 0$$

is a trinomial equation of the second degree.

165 b. What are the factors of the trinomial equation

$$x^2 - 4x - 12 = 0?$$

A trinomial equation of the second degree may always be reduced to one of the four forms (§ 160) by simply transposing the known term to the second member, and then solving the equation. Thus, from the above equation we have

$$x^2 - 4x = 12.$$

Resolving the equation, we find the two roots to be $+6$ and -2: therefore the factors are $x-6$ and $x+2$ (§ 162).

Since the sum of the two roots is equal to the coefficient of the second term taken with a contrary sign (§ 163), and the product of the two roots is equal to the known term in the second member taken with a contrary sign, or to the third term of the trinomial taken with the *same sign*, it follows that any trinomial may be factored by inspection when two numbers can be discovered *whose algebraic sum is equal to the coefficient of the second term, and whose product is equal to the third term.*

Exercises.

1. What are the factors of the trinomial $x^2-9x-36$?

It is seen by inspection that $+12$ and -3 will fulfill the conditions of roots: for $12-3=9$, that is, the coefficient of the second term with a contrary sign; and $12\times -3=-36$, the third term of the trinomial. Hence the factors are $x-12$ and $x+3$.

2. What are the factors of $x^2-7x-30=0$?
Ans. $x-10$ and $x+3$.

3. What are the factors of $x^2+15x+36=0$?
Ans. $x+12$ and $x+3$.

4. What are the factors of $x^2-12x-28=0$?
Ans. $x-14$ and $x+2$.

5. What are the factors of $x^2-7x-8=0$?
Ans. $x-8$ and $x+1$.

In the trinomial equation of the form $x^{2n}+2px^n=q$, the exponent of x in the first term is double the exponent of x in the second term.

$$x^6-4x^3=32, \text{ and } x^4+4x^2=117,$$

are both equations of this form, and may be solved by the

rules already given for the solution of equations of the second degree.

In the equation $x^{2n} + 2px^n = q$

we see that the first member will become a perfect square by adding to it the square of half the coefficient of x^n. Thus,

$$x^{2n} + 2px^n + p^2 = q + p^2,$$

in which the first member is a perfect square. Then, extracting the square root of both members, we have

$$x^n + p = \pm\sqrt{q + p^2}.$$

Hence $$x^n = -p \pm \sqrt{q + p^2}.$$

By taking the nth root of both members,

$$x' = \sqrt[n]{-p + \sqrt{q + p^2}},$$

and $$x'' = \sqrt[n]{-p - \sqrt{-p + p^2}}.$$

Exercises.

1. What are the values of x in the equation

$$x^6 + 6x^3 = 112?$$

Completing the square,

$$x^6 + 6x^3 + 9 = 112 + 9 = 121.$$

Extracting the square root of both members,

$$x^3 + 3 = \pm\sqrt{121} = \pm 11.$$

Hence $x' = \sqrt[3]{-3 + 11}$, and $x'' = \sqrt[3]{-3 - 11}$.

Hence $x' = \sqrt[3]{8} = 2$, and $x'' = \sqrt[3]{-14} = -\sqrt[3]{14}$.

2. What are the values of x in the equation $x^4 - 8x^2 = 9$?

Completing the square, we have

$$x^4 - 8x^2 + 16 = 9 + 16 = 25.$$

Extracting the square root of both members,

$$x^2 - 4 = \pm\sqrt{25} = \pm 5.$$

Hence $\qquad x' = \pm\sqrt{4+5}$, and $x'' = \pm\sqrt{4-5}$.

Hence $\qquad x' = +3$ and -3; and $x'' = +\sqrt{-1}$ and $-\sqrt{-1}$.

3. What are the values of x in the equation $x^6 + 20x^3 = 69$?

Completing the square,

$$x^6 + 20x^3 + 100 = 69 + 100 = 169.$$

Extracting the square root of both members,

$$x^3 + 10 = \pm\sqrt{169} = \pm 13.$$

Hence $\qquad x' = \sqrt[3]{-10+13}$, and $x'' = \sqrt[3]{-10-13}$.

Hence $\qquad x' = \sqrt[3]{3}$, and $x'' = \sqrt[3]{-23}$.

4. What are the values of x in the equation $x^4 - 2x^2 = 3$?

Ans. $x' = \pm\sqrt{3}$, and $x'' = \pm\sqrt{-1}$.

5. What are the values of x in the equation $x^6 + 8x^3 = 9$?

Ans. $x' = 1$, and $x'' = \sqrt[3]{-9}$.

6. Given $x \pm \sqrt{9x+4} = 12$, to find x.

Transposing x to the second member, and squaring,

$$9x + 4 = x^2 - 24x + 144.$$

$$\therefore\ x^2 - 33x = -140,$$

and $\qquad x' = 28$, and $x'' = 5$.

7. $4x \pm 4\sqrt{x+2} = 7$. *Ans.* $x' = 4\frac{1}{4}$, $x'' = \frac{1}{4}$.

8. $x \pm \sqrt{5x+10} = 8$. *Ans.* $x' = 18$, $x'' = 3$.

NUMERICAL VALUES OF THE ROOTS.

166. We have seen (§ 160) that, by attributing all possible signs to $2p$ and q, we have the four following forms: —

$$x^2 + 2px = q \tag{1}$$

$$x^2 - 2px = q \tag{2}$$

$$x^2 + 2px = -q \tag{3}$$

$$x^2 - 2px = -q \tag{4}$$

First Form.

167. Since q is positive, we know, from the Fourth Property, that the product of the roots must be negative: hence *the roots have contrary signs.* Since the coefficient $2p$ is positive, we know, from the Third Property, that the algebraic sum of the roots is negative: hence *the negative root is numerically the greater.*

Second Form.

168. Since q is positive, the product of the roots must be negative: hence *the roots have contrary signs.* Since $2p$ is negative, the algebraic sum of the roots must be positive: hence *the positive root is numerically the greater.*

Third Form.

169. Since q is negative, the product of the roots is positive (Fourth Property): hence *the roots have the same sign.* Since $2p$ is positive, the sum of the roots must be negative: hence *both are negative.*

Fourth Form.

170. Since q is negative, the product of the roots is positive: hence *the roots have the same sign.* Since $2p$ is negative, the sum of the roots is positive: hence *the roots are both positive.*

171. If we make $q = 0$, the first form becomes

$$x^2 + 2px = 0, \text{ or } x(x + 2p) = 0;$$

which shows that one root is equal to 0, and the other to $-2p$.

Under the same supposition, the second form becomes

$$x^2 - 2px = 0, \text{ or } x(x - 2p) = 0;$$

which shows that one root is equal to 0, and the other to $2p$. Both of these results are as they should be; since, when q, the product of the roots, becomes 0, one of the factors must be 0, and hence one root must be 0.

172. If, in the Third and Fourth Forms, $q > p^2$, the quantity under the radical sign will become *negative:* hence *its square root cannot be extracted* (§ 137). Under this supposition, the values of x are *imaginary.* How are these results to be interpreted?

If a given number be divided into two parts, their product will be the greatest possible when the parts are equal.

Denote the number by $2p$, and the difference of the parts by d. Then

$$p + \frac{d}{2} = \text{the greater part (p. 114),}$$

$$p - \frac{d}{2} = \text{the less part,}$$

and $$p^2 - \frac{d^2}{4} = P, \text{ their product.}$$

It is plain that the product P will *increase* as d *diminishes,* and that it will be the *greatest possible* when $d = 0$, for then there will be no negative quantity to be subtracted from p^2 in the first member of the equation. But when $d = 0$, the parts are equal: hence *the product of the two parts is the greatest when they are equal.*

In the equations

$$x^2+2px=-q, \quad x^2-2px=-q,$$

$2p$ is the sum of the roots, and $-q$ their product; and hence, by the principle just established, the product q can never be greater than p^2. This condition fixes a limit to the value of q. If, then, we make $q>p^2$, we pass this limit, and express by the equation a condition which cannot be fulfilled; and this incompatibility of the conditions is made apparent by the values of x becoming imaginary. Hence we conclude that

When the values of the unknown quantity are imaginary, the conditions of the proposition are incompatible with each other.

PROBLEMS FOR SOLUTION.

1. Find two numbers whose sum shall be 12, and product 46.

Let $\quad x$ and $y=$ the numbers.

By the 1st condition, $\quad x+y=12,$

and by the 2d, $\quad xy=46.$

The first equation gives $\quad x=12-y.$

Substituting this value for x in the second, we have

$$12y-y^2=46.$$

Changing the signs of the terms, we have

$$y^2-12y=-46.$$

Completing the square,

$$y^2-12y+36=-46+36=-10,$$

which gives $\quad y'=6+\sqrt{-10},$

and $\quad y''=6-\sqrt{-10};$

both of which values are imaginary, as indeed they should be, since the conditions are incompatible.

2. The sum of two numbers is 8, and their product 20. What are the numbers?

Let $\qquad x$ and $y =$ the numbers.

By the 1st condition, $\qquad x + y = 8.$

By the 2d, $\qquad xy = 20.$

The first equation gives $\qquad x = 8 - y.$

Substituting this value of x in the second, we have

$$8y - y^2 = 20.$$

Changing the signs, and completing the square, we have

$$y^2 - 8y + 16 = -4.$$

Extracting the root,

$$y' = 4 + \sqrt{-4}, \text{ and } y'' = 4 - \sqrt{-4}.$$

These values of y may be put under the forms (§ 142),

$$y = 4 + 2\sqrt{-1}, \text{ and } y = 4 - 2\sqrt{-1}.$$

3. What are the values of x in the equation $x^2 + 2x = -10$?

$$Ans. \begin{cases} x' = -1 + 3\sqrt{-1}. \\ x'' = -1 - 3\sqrt{-1}. \end{cases}$$

4. Find a number such that twice its square added to three times the number shall give 65.

Let $\qquad x =$ unknown number.

Then the equation of the problem is

$$2x^2 + 3x = 65.$$

Whence $\qquad x = -\frac{3}{4} \pm \sqrt{\frac{65}{2} + \frac{9}{16}} = -\frac{3}{4} \pm \frac{23}{4}.$

$$\therefore\ x' = -\frac{3}{4} + \frac{23}{4} = 5, \text{ and } x'' = -\frac{3}{4} - \frac{23}{4} = -\frac{13}{2}.$$

Both these values satisfy the equation of the problem: for

$$2 \times (5)^2 + 3 \times 5 = 2 \times 25 + 15 = 65,$$

and $\qquad 2\left(-\frac{13}{2}\right)^2 + 3 \times -\frac{13}{2} = \frac{169}{2} - \frac{39}{2} = \frac{130}{2} = 65.$

NOTES. — 1. If we restrict the enunciation of the problem to its arithmetical sense, in which "added" means "numerical increase," the first value of x only will satisfy the conditions of the problem.

2. If we give to "added" its algebraical signification (when it may mean subtraction as well as addition), the problem may be thus stated: —

To find a number such that twice its square diminished by three times the number shall give 65.

The second value of x will satisfy this enunciation; for

$$2\left(\frac{13}{2}\right)^2 - 3 \times \frac{13}{2} = \frac{169}{2} - \frac{39}{2} = 65.$$

3. The root which results from giving the plus sign to the radical is generally an answer to the question in its arithmetical sense. The second root generally satisfies the problem under a modified statement.

Thus, in the example it was required to find a number of which twice the square *added* to three times the number shall give 65. Now, in the arithmetical sense, "added" means increased; but in the algebraic sense it implies diminution when the quantity added is negative. In this sense, the second root satisfies the enunciation.

5. A certain person purchased a number of yards of cloth for 240 cents. If he had purchased 3 yards *less* of the same cloth for the same sum, it would have cost him 4 cents more per yard. How many yards did he buy?

Let $\qquad x =$ number of yards purchased.

Then $\qquad \dfrac{240}{x} =$ the price per yard.

If for 240 cents he had purchased three yards less, that is, $x - 3$ yards, the price per yard, under this hypothesis, would have been denoted by $\dfrac{240}{x-3}$. But by the conditions this last cost must exceed the first by 4 cents. Therefore we have the equation

$$\frac{240}{x-3} - \frac{240}{x} = 4.$$

Whence $\qquad x^2 - 3x = 180,$

and $\qquad x = \frac{3}{2} \pm \sqrt{\frac{9}{4} + 180} = \frac{3 \pm 27}{2}.$

$$\therefore x' = 15, \text{ and } x'' = -12.$$

NOTES. — 1. The value $x' = 15$ satisfies the enunciation in its arithmetical sense; for if 15 yards cost 240 cents, $240 \div 15 = 16$ cents, the price of one yard; and $240 \div 12 = 20$ cents, the price of 1 yard under the second supposition.

2. The second value of x is an answer to the following problem: —

A certain person purchased a number of yards of cloth for 240 cents. If he had paid the same for three yards *more*, it would have cost him 4 cents *less* per yard. How many yards did he buy?

This would give the equation of condition

$$\frac{240}{x} - \frac{240}{x+3} = 4,$$

or

$$x^2 - 3x = 180,$$

the same equation as found before. Hence

A single equation will often state two or more arithmetical problems.

This arises from the fact that the language of algebra is more comprehensive than that of arithmetic.

6. A man, having bought a horse, sold it for \$24. At the sale he lost as much per cent on the price of the horse as the horse cost him dollars. What did he pay for the horse?

Let $x =$ the number of dollars that he paid for the horse. Then $x - 24 =$ the loss he sustained. But as he lost x per cent by the sale, he must have lost $\frac{x}{100}$ upon each dollar, and upon x dollars he lost a sum denoted by $\frac{x^2}{100}$. We have, then, the equation

$$\frac{x^2}{100} = x - 24, \text{ whence } x^2 - 100x = -2400,$$

and

$$x = 50 \pm \sqrt{2500 - 2400} = 50 \pm 10.$$

$$\therefore\ x' = 60, \text{ and } x'' = 40.$$

Both of these roots will satisfy the problem: for, if the man gave \$60 for the horse, and sold him for \$24, he lost \$36. From the enunciation he should have lost 60 per cent of \$60; that is,

$$\frac{60}{100} \text{ of } 60 = \frac{60 \times 60}{100} = 36.$$

Therefore \$60 satisfies the enunciation.

Had he paid \$40 for the horse, he would have lost by the sale \$16. From the enunciation he should have lost 40 per cent of \$40; that is,

$$\frac{40}{100} \text{ of } 40 = \frac{40 \times 40}{100} = 16.$$

Therefore \$40 satisfies the enunciation.

7. The sum of two numbers is 11, and the sum of their squares is 61. What are the numbers? *Ans.* 5 and 6.

8. The difference of two numbers is 3, and the sum of their squares is 89. What are the numbers? *Ans.* 5 and 8.

9. A grazier bought as many sheep as cost him £60; and, after reserving fifteen out of the number, he sold the remainder for £54, and gained 2s. a head on those he sold. How many did he buy? *Ans.* 75.

10. A merchant bought cloth, for which he paid £33 15s., which he sold again at £2 8s. per piece, and gained by the bargain as much as one piece cost him. How many pieces did he buy? *Ans.* 15.

11. The difference of two numbers is 9, and their sum multiplied by the greater is equal to 266. What are the numbers? *Ans.* 14 and 5.

12. Find a number such that, if you subtract it from 10 and multiply the remainder by the number itself, the product will be 21. *Ans.* 7 or 3.

13. A person traveled 105 miles. If he had traveled 2 miles an hour slower, he would have been 6 hours longer in completing the same distance. How many miles did he travel per hour? *Ans.* 7 miles.

14. A person purchased a number of sheep, for which he paid \$224. Had he paid for each twice as much, plus

\$2, the number bought would have been represented by twice what was paid for each. How many sheep were purchased? *Ans.* 32.

15. The difference of two numbers is 7, and their sum multiplied by the greater is equal to 130. What are the numbers? *Ans.* 10 and 3.

16. Divide 100 into two such parts that the sum of their squares shall be 5392. *Ans.* 64 and 36.

17. Two square courts are paved with stones a foot square. The larger court is 12 feet larger than the smaller one, and the number of stones in both pavements is 2120. How long is the smaller pavement? *Ans.* 26 feet.

18. Two hundred and forty dollars are equally distributed among a certain number of persons. The same sum is again distributed amongst a number greater by 4. In the latter case each receives ten dollars less than in the former. How many persons were there in each case? *Ans.* 8 and 12.

19. Two partners, A and B, gained \$360. A's money was in trade 12 months, and he received for principal and profit \$520. B's money was \$600, and was in trade 16 months. How much capital had A? *Ans.* \$400.

Equations containing more than One Unknown Quantity.

173. Two simultaneous equations, each of the second degree, and containing two unknown quantities, will, when combined, generally give rise to an equation of the fourth degree. Hence only particular cases of such equations can be solved by the methods already given.

CASE I.

Two simultaneous equations, involving two unknown quantities, may readily be solved when one is of the first and the other of the second degree.

(1) Given $\begin{Bmatrix} x + y = 14 \\ x^2 + y^2 = 100 \end{Bmatrix}$, to find x and y.

Ans. $\begin{cases} x' = 8, & x'' = 6. \\ y' = 6, & y'' = 8. \end{cases}$

Transposing y in the first equation, we have

$$x = 14 - y.$$

Squaring both members, $x^2 = 196 - 28y + y^2.$

Substituting this value for x^2 in the second equation, we have

$$196 - 28y + y^2 + y^2 = 100,$$

from which we have $y^2 - 14y = -48.$

Completing the square,

$$y^2 - 14y + 49 = 1.$$

Extracting the square root,

$$y - 7 = \pm\sqrt{1} = +1 \text{ and } -1.$$

Hence $y' = 7 + 1 = 8,$

and $y'' = 7 - 1 = 6.$

If we take the greater value, we find $x = 6$; and if we take the lesser, we find $x = 8$.

VERIFICATION. For the greater value, $y = 8$, the equation

$$x + y = 14$$

gives $6 + 8 = 14,$

and $x^2 + y^2 = 100$

gives $36 + 64 = 100.$

For the value $y = 6$, the equation

$$x + y = 14$$

gives $$8 + 6 = 14,$$

and $$x^2 + y^2 = 100$$

gives $$64 + 36 = 100.$$

Hence both sets of values satisfy the given equation.

(2) Given $\left\{\begin{array}{l} x - y = 3 \\ x^2 - y^2 = 45 \end{array}\right\}$, to find x and y.

Transposing y in the first equation, we have

$$x = 3 + y.$$

Squaring both members, $$x^2 = 9 + 6y + y^2.$$

Substituting this value for x^2 in the second equation, we have

$$9 + 6y + y^2 - y^2 = 45.$$

Whence $$6y = 36, \text{ and } y = 6.$$

Substituting this value of y in the first equation, we have

$$x - 6 = 3,$$

and consequently $$x' = 3 + 6 = 9.$$

VERIFICATION.

$$x - y = 3 \text{ gives } 9 - 6 = 3,$$

and $$x^2 - y^2 = 45 \text{ gives } 81 - 36 = 45.$$

Exercises.

Solve the following simultaneous equations: —

1. $\left\{\begin{array}{l} x + y = 12 \\ x^2 - y^2 = 24 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x' = 7. \\ y' = 5. \end{array}\right.$

2. $\left\{\begin{array}{l} x - y = 3 \\ x^2 + y^2 = 117 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x' = 9,\ x'' = -6. \\ y' = 6,\ y'' = -9. \end{array}\right.$

3. $\left\{\begin{array}{l} x + y = 9 \\ x^2 - 2xy + y^2 = 1 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x' = 5,\ x'' = 5. \\ y' = 4,\ y'' = 4. \end{array}\right.$

4. $\left\{\begin{array}{l} x - y = 5 \\ x^2 + 2xy + y^2 = 225 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x' = 10,\ x'' = -5. \\ y' = 5,\ y'' = -10. \end{array}\right.$

CASE II.

174. *Two simultaneous equations of the second degree, which are homogeneous with respect to the unknown quantity, may readily be solved.*

Given $$\begin{cases} x^2 + 3xy = 22 & (1) \\ x^2 + 3xy + 2y^2 = 40 & (2) \end{cases}$$

to find x and y.

Assume $x = ty$, t being any auxiliary unknown quantity.

Substituting this value of x in Equations (1) and (2), we have

$$t^2y^2 + 3ty^2 = 22 \qquad \therefore\ y^2 = \frac{22}{t^2 + 3t} \qquad (3)$$

$$t^2y^2 + 3ty^2 + 2y^2 = 40 \qquad \therefore\ y^2 = \frac{40}{t^2 + 3t + 2} \qquad (4)$$

Hence $$\frac{22}{t^2 + 3t} = \frac{40}{t^2 + 3t + 2}.$$

Hence $$22t^2 + 66t + 44 = 40t^2 + 120t.$$

Reducing, $$t^2 + 3t = \frac{22}{9}.$$

Whence, $$t' = \frac{2}{3}, \text{ and } t'' = -\frac{11}{3}.$$

Substituting either of these values in Equations (3) or (4), we find

$$y' = +3, \text{ and } y'' = -3.$$

Substituting the plus value of y in Equation (1), we have

$$x^2 + 9x = 22,$$

from which we find

$$x' = +2, \text{ and } x'' = -11.$$

If we take the negative value, $y'' = -3$, we have from Equation (1)

$$x^2 - 9x = 22,$$

from which we find $x' = +11$, and $x'' = -2$.

VERIFICATION. For the values $y' = +3$, and $x' = +2$, the given equation

$$x^2 + 3xy = 22,$$

gives $$2^2 + 3 \times 2 \times 3 = 4 + 18 = 22;$$

and for the second value, $x'' = -11$, the same equation,

$$x^2 + 3xy = 22,$$

gives $$(-11)^2 + 3 \times -11 \times 3 = 121 - 99 = 22.$$

If, now, we take the second value of y (that is, $y'' = -3$) and the corresponding values of x (viz., $x' = +11$, and $x'' = -2$) for $x' = +11$, the given equation,

$$x^2 + 3xy = 22,$$

gives $$11^2 + 3 \times 11 \times -3 = 121 - 99 = 22;$$

and for $x'' = -2$, the same equation,

$$x^2 + 3xy = 22,$$

gives $$(-2)^2 + 3 \times -2 \times -3 = 4 + 18 = 22.$$

The verifications could be made in the same way by employing Equation (2).

NOTE. — In equations similar to the above we generally find but a single pair of values, corresponding to the values in this equation of $y' = +3$, and $x' = +2$. The complete solution would give four pairs of values.

Exercises.

Find the values of x and y in the following equations: —

1. $\begin{Bmatrix} x^2 - y^2 = -9 \\ y^2 - xy = 5 \end{Bmatrix}$. *Ans.* $\begin{cases} x = 4. \\ y = 5. \end{cases}$

2. $\begin{Bmatrix} xy - y^2 = -7 \\ y^2 + x^2 = 85 \end{Bmatrix}$. *Ans.* $\begin{cases} x = 6. \\ y = 7. \end{cases}$

3. $\begin{Bmatrix} 2x^2 + 3xy = 470 \\ y^2 - xy = -9 \end{Bmatrix}$. *Ans.* $\begin{cases} x = 10. \\ y = 9. \end{cases}$

4. $\begin{Bmatrix} 5xy - 3y^2 = 32 \\ x^2 + y^2 + 3xy = 71 \end{Bmatrix}$. *Ans.* $\begin{cases} x = 7. \\ y = 1. \end{cases}$

CASE III.

175. Many other equations of the second degree may be so transformed as to be brought under the rules of solution already given. The seven formulas following will aid in such transformation.

Formula 1. — When the sum and the difference are known.

$$x+y=s.$$

$$x-y=d.$$

Then [p. 125, (3)], $\quad x=\frac{s+d}{2}=\frac{1}{2}s+\frac{1}{2}d,$

and $\quad y=\frac{s-d}{2}=\frac{1}{2}s-\frac{1}{2}d.$

Formula 2. — When the sum and the product are known.

$$x+y=s \tag{1}$$

$$xy=p \tag{2}$$

Squaring (1), $\quad x^2+2xy+y^2=s^2$
Multiplying (2) by 4, $\quad 4xy \quad =4p$
Subtracting, $\quad x^2-2xy+y^2=s^2-4p$

Extracting root, $\quad x-y=\pm\sqrt{s^2-4p}.$

But $\quad x+y=s.$

Hence $\quad x=\frac{s}{2}\pm\frac{1}{2}\sqrt{s^2-4p},$

and $\quad y=\frac{s}{2}\mp\frac{1}{2}\sqrt{s^2-4p}.$

Formula 3. — When the difference and the product are known.

$$x - y = d \quad (1)$$

$$xy = p \quad (2)$$

Squaring (1), $x^2 - 2xy + y^2 = d^2$

Multiplying (2) by 4, $4xy = 4p$

Adding, $x^2 + 2xy + y^2 = d^2 + 4p$

$$x + y = \pm\sqrt{d^2 + 4p}$$

$$x - y = d$$

$$x = \frac{1}{2}d \pm \frac{1}{2}\sqrt{d^2 + 4p}$$

$$y = -\frac{1}{2}d \pm \frac{1}{2}\sqrt{d^2 + 4p}.$$

Formula 4. — When the sum of the squares and the product are known.

$$x^2 + y^2 = s \quad (1)$$

$$xy = p \quad (2)$$

$$\therefore\ 2xy = 2p \quad (3)$$

Adding (1) and (3),

$$x^2 + 2xy + y^2 = s + 2p.$$

Hence $x + y = \pm\sqrt{s + 2p}$ (4)

Subtracting (3) from (1),

$$x^2 - 2xy + y^2 = s - 2p.$$

Hence $x - y = \pm\sqrt{s - 2p}$ (5)

Combining (4) and (5), $x = \frac{1}{2}\sqrt{s + 2p} + \frac{1}{2}\sqrt{s - 2p}$,

and $y = \frac{1}{2}\sqrt{s + 2p} - \frac{1}{2}\sqrt{s - 2p}$.

Formula 5. — When the sum, and the sum of the squares, are known.

$$x + y = s \qquad (1)$$

$$x^2 + y^2 = s' \qquad (2)$$

Squaring (1), $$x^2 + 2xy + y^2 = s^2$$

$$2xy = s^2 - s'$$

$$xy = \frac{s^2 - s'}{2} = p. \qquad (3)$$

By putting $xy = p$, and combining Equations (1) and (3) by Formula 2, we find the values of x and y.

Formula 6. — When the sum, and the sum of the cubes, are known.

$$x + y = 8 \qquad (1)$$

$$x^3 + y^3 = 152 \qquad (2)$$

Cubing (1), $$x^3 + 3x^2y + 3xy^2 + y^3 = 512$$

Subtracting, $$3x^2y + 3xy^2 = 360$$

Factoring, $$3xy(x + y) = 360$$

From (1), $$3xy(8) = 360$$

$$24xy = 360$$

Hence $$xy = 15 \qquad (3)$$

Combining (1) and (3), we find $x = 5$, and $y = 3$.

Formula 7. — When we have an equation of the form

$$(x + y)^2 + (x + y) = q.$$

Let us assume $$x + y = z.$$

Then the given equation becomes

$$z^2 + z = q, \text{ and } z = -\frac{1}{2} \pm \sqrt{q + \frac{1}{4}}.$$

$$x + y = -\frac{1}{2} \pm \sqrt{q + \frac{1}{4}}.$$

(1) Given $\left\{\begin{array}{rr} xz = y^2 & (1) \\ x + y + z = 7 & (2) \\ x^2 + y^2 + z^2 = 21 & (3) \end{array}\right\}$, to find x, y, and z.

Transposing y in (2), we have

$$x + z = 7 - y \tag{4}$$

Squaring the members, we have

$$x^2 + 2xz + z^2 = 49 - 14y + y^2.$$

If, now, we substitute for $2xz$ its value taken from (1), we have

$$x^2 + 2y^2 + z^2 = 49 - 14y + y^2.$$

Canceling y^2 in each member, there results

$$x^2 + y^2 + z^2 = 49 - 14y.$$

But from (3) we see that each member of the last equation is equal to 21. Hence

$$49 - 14y = 21,$$

and $$14y = 49 - 21 = 28.$$

Hence $$y = \frac{28}{14} = 2.$$

Substituting this value of y in (1) gives

$$xz = 4.$$

Substituting it in (4) gives

$$x + z = 5, \text{ or } x = 5 - z.$$

Substituting this value of x in the previous equation, we obtain

$$5z - z^2 = 4, \text{ or } z^2 - 5z = -4.$$

Completing the square, we have

$$z^2 - 5z + 6.25 = 2.25;$$

and $$z - 2.5 = \pm\sqrt{2.25} = +1.5 \text{ or } -1.5.$$

Hence $$z = 2.5 + 1.5 = 4, \text{ and } z = +2.5 - 1.5 = 1.$$

(2) Given $\left\{\begin{array}{ll} x + \sqrt{xy} + y = 19 & (1) \\ x^2 + xy + y^2 = 133 & (2) \end{array}\right\}$, to find x and y.

Dividing (2) by (1), we have

$$x - \sqrt{xy} + y = 7$$

But $$x + \sqrt{xy} + y = 19$$

Hence, by addition, $$2x + 2y = 26$$

or $$x + y = 13$$

Substituting in (1), $$\sqrt{xy} + 13 = 19$$

or, transposing, $$\sqrt{xy} = 6$$

Squaring, $$xy = 36.$$

(2) is $$x^2 + xy + y^2 = 133$$

and from the last we have $$3xy = 108$$

Subtracting, $$x^2 - 2xy + y^2 = 25$$

Hence $$x - y = \pm 5$$

But $$x + y = 13$$

Hence $$x = 9 \text{ and } 4, \text{ and } y = 4 \text{ and } 9.$$

PROBLEMS FOR SOLUTION.

1. Find two numbers such that their sum shall be 15, and the sum of their squares 113.

Let x and y = the numbers.

Then $$x + y = 15 \qquad (1)$$

and $$x^2 + y^2 = 113 \qquad (2)$$

From (1) we have $$x^2 = 225 - 30y + y^2.$$

Substituting this value in (2),

$$225 - 30y + y^2 + y^2 = 113.$$

Hence $$2y^2 - 30y = -112,$$

and $$y^2 - 15y = -56.$$

Hence $$y' = 8, \text{ and } y'' = 7.$$

The first value of y being substituted in (1) gives $x' = 7$; and the second, $x'' = 8$. Hence the numbers are 7 and 8.

2. Find two numbers such that their product added to their sum shall be 17, and their sum taken from the sum of their squares shall leave 22.

Let x and y = the numbers.

Then, from the conditions,

$$(x+y)+xy=17 \quad (1)$$

$$x^2+y^2-(x+y)=22 \quad (2)$$

Multiplying (1) by 2, we have

$$2xy+2(x+y)=34 \quad (3)$$

Adding (2) and (3), we have

$$x^2+2xy+y^2+(x+y)=56.$$

Hence
$$(x+y)^2+(x+y)=56 \quad (4)$$

Regarding $(x+y)$ as a single unknown quantity,

$$x+y=-\frac{1}{2}\pm\sqrt{56+\frac{1}{4}}=7.$$

Substituting this value in (1), we have

$$7+xy=17, \text{ and } y=5.$$

Hence the numbers are 2 and 5.

3. What two numbers are those whose sum is 8, and sum of their squares 34? *Ans.* 5 and 3.

4. It is required to find two such numbers that the first shall be to the second as the second is to 16, and the sum of whose squares shall be 225. *Ans.* 9 and 12.

5. What two numbers are those which are to each other as 3 to 5, and whose squares added together make 1666? *Ans.* 21 and 35.

6. There are two numbers whose difference is 7, and half their product plus 30 is equal to the square of the less number. What are the numbers? *Ans.* 12 and 19.

7. What two numbers are those whose sum is 5, and the sum of their cubes 35? *Ans.* 2 and 3.

8. What two numbers are those whose sum is to the greater as 11 to 7, and the difference of whose squares is 132?

Ans. 14 and 8.

9. Divide the number 100 into two such parts that the product may be to the sum of their squares as 6 to 13.

Ans. 40 and 60.

10. Two persons, A and B, departed from different places at the same time, and traveled towards each other. On meeting, it appeared that A had traveled 18 miles more than B, and that A could have gone B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. How far did each travel? *Ans.* A, 72 miles; B, 54 miles.

11. There are two numbers whose difference is 15, and half their product is equal to the cube of the lesser number. What are those numbers? *Ans.* 3 and 18.

12. What two numbers are those whose sum multiplied by the greater is equal to 77, and whose difference multiplied by the less is equal to 12? *Ans.* 4 and 7, or $\frac{3}{2}\sqrt{2}$ and $\frac{11}{2}\sqrt{2}$.

13. Divide 100 into two such parts that the sum of their square roots may be 14. *Ans.* 64 and 36.

14. It is required to divide the number 24 into two such parts that their product may be equal to 35 times their difference. *Ans.* 10 and 14.

15. The sum of two numbers is 8, and the sum of their cubes is 152. What are the numbers? *Ans.* 3 and 5.

16. Two merchants each sold the same kind of stuff. The second sold 3 yards more of it than the first, and together

they receive \$35. The first said to the second, "I would have received \$24 for your stuff." The other replied, "And I should have received \$12½ for yours." How many yards did each of them sell?

Ans. 1st merchant, 15 or 5; 2d merchant, 18 or 8.

17. A widow possessed \$13,000, which she divided into two parts, and placed them at interest in such a manner that the incomes from them were equal. If she had put out the first portion at the same rate as the second, she would have drawn for this part \$360 interest; and if she had placed the second out at the same rate as the first, she would have drawn for it \$490 interest. What were the two rates of interest?

Ans. 7 and 6 per cent.

18. Find three numbers such that the difference between the third and second shall exceed the difference between the second and first by 6, that the sum of the numbers shall be 33, and the sum of their squares 467. *Ans.* 5, 9, and 19.

19. What number is that which being divided by the product of its two digits the quotient will be 3, and if 18 be added to it the resulting number will be expressed by the digits inverted? *Ans.* 24.

20. What two numbers are those which are to each other as m to n, and the sum of whose squares is b?

$$Ans.\ \frac{m\sqrt{b}}{\sqrt{m^2+n^2}},\ \frac{n\sqrt{b}}{\sqrt{m^2+n^2}}.$$

21. What two numbers are those which are to each other as m to n, and the difference of whose squares is b?

$$Ans.\ \frac{m\sqrt{b}}{\sqrt{m^2-n^2}},\ \frac{n\sqrt{b}}{\sqrt{m^2-n^2}}.$$

22. Required to find three numbers such that the product of the first and second shall be equal to 2, the product of the

first and third equal to 4, and the sum of the squares of the second and third equal to 20. *Ans.* 1, 2, and 4.

23. It is required to find three numbers whose sum shall be 38, the sum of their squares 634, and the difference between the second and first greater by 7 than the difference between the third and second. *Ans.* 3, 15, and 20.

24. Required to find three numbers such that the product of the first and second shall be equal to a, the product of the first and third equal to b, and the sum of the squares of the second and third equal to c.

Ans. $\sqrt{\frac{a^2+b^2}{c}},\ a\sqrt{\frac{c}{a^2+b^2}},\ b\sqrt{\frac{c}{a^2+b^2}}.$

25. What two numbers are those whose sum multiplied by the greater gives 144, and whose difference multiplied by the less gives 14? *Ans.* 9 and 7.

CHAPTER IX.

PROPORTIONS AND PROGRESSIONS.

176. Two quantities of the same kind may be compared, the one with the other, in two ways, — first, by considering *how much* one is greater or less than the other, which is shown by their difference; and, second, by considering *how many times* one is greater or less than the other, which is shown by their quotient.

Thus, in comparing the numbers 3 and 12 together with respect to their difference, we find that 12 *exceeds* 3 by 9; and in comparing them together with respect to their quotient, we find that 12 *contains* 3 four times, or that 12 is four times as great as 3.

The first of these methods of comparison is called **arithmetical proportion**; and the second, **geometrical proportion.** Hence

Arithmetical proportion considers the relation of quantities with respect to their difference; and geometrical proportion, the relation of quantities with respect to their quotient.

Arithmetical Proportion and Progression.

177. If we have four numbers, 2, 4, 8, and 10, of which the difference between the first and second is equal to the difference between the third and fourth, these numbers are said to be in arithmetical proportion. The first term, 2, is called an **antecedent**; and the second term, 4, with which it is compared, a **consequent.** The number 8 is also called an antecedent; and the number 10, with which it is compared, a consequent.

When the difference between the first and second is equal to the difference between the third and fourth, the four numbers are said to be in proportion. Thus, the numbers

$$2, \quad 4, \quad 8, \quad 10,$$

are in arithmetical proportion.

178. When the difference between the first antecedent and consequent is the same as between any two consecutive terms of the proportion, the proportion is called an **arithmetical progression.** Hence a *progression by differences*, or an *arithmetical progression*, is a series in which the successive terms are continually increased or decreased by a constant number, which is called the **common difference** of the progression.

A **series** is a succession of terms, each of which is derived from one or more of the preceding ones by a *fixed law*, called the **law of the series.**

In the two series,

$$1, \quad 4, \quad 7, \quad 10, \quad 13, \quad 16, \quad 19, \quad 22, \quad 25, \quad \ldots\ldots$$
$$60, \quad 56, \quad 52, \quad 48, \quad 44, \quad 40, \quad 36, \quad 32, \quad 28, \quad \ldots\ldots$$

the first is called an **increasing progression,** of which the common difference is 3; and the second, a **decreasing progression,** of which the common difference is 4.

In general, let a, b, c, d, e, f, ... denote the terms of a progression by differences. It has been agreed to write them thus: —

$$a \,.\, b \,.\, c \,.\, d \,.\, e \,.\, f \,.\, g \,.\, h \,.\, i \,.\, k \ldots\ldots$$

This series is read, "a is to b, as b is to c, as c is to d, as d is to e," etc. This is a series of *continued equi-differences*, in which each term is at the same time an antecedent and a consequent, with the exception of the first term, which is only an *antecedent*, and the last, which is only a consequent.

179. Let d denote the common difference of the progression,

$$a \,.\, b \,.\, c \,.\, e \,.\, f \,.\, g \,.\, h, \text{ etc.},$$

which we will consider increasing.

From the definition of the progression, it evidently follows that

$$b = a + d, \quad c = b + d = a + 2d, \quad e = c + d = a + 3d;$$

and, in general, any term of the series is equal to *the first term, plus as many times the common difference as there are preceding terms.*

Thus, let l be any term, and n the number which marks the place of it. The expression for this *general term* is

$$l = a + (n - 1)\,d.$$

Hence, for finding the last term, we have the following rule: —

Multiply the common difference by the number of terms less one.

To the product add the first term. The sum will be the last term.

The formula $\qquad l = a + (n - 1)\,d$

serves to find any term whatever, without determining all those which precede it.

Exercises.

1. If we make $n = 1$, we have $l = a$; that is, the series will have but one term.

2. If we make $n = 2$, we have $l = a + d$; that is, the series will have two terms, and the second term is equal to *the first*, plus the common difference.

3. If $a=3$, and $d=2$, what is the 3d term? *Ans.* 7.

4. If $a=5$, and $d=4$, what is the 6th term? *Ans.* 25.

5. If $a=7$, and $d=5$, what is the 9th term? *Ans.* 47.

6. If $a=8$, and $d=5$, what is the 10th term? *Ans.* 53.

7. If $a=20$, and $d=4$, what is the 12th term? *Ans.* 64.

8. If $a=40$, and $d=20$, what is the 50th term?
Ans. 1020.

9. If $a=45$, and $d=30$, what is the 40th term?
Ans. 1215.

10. If $a=30$, and $d=20$, what is the 60th term?
Ans. 1210.

11. If $a=50$, and $d=10$, what is the 100th term?
Ans. 1040.

12. To find the 50th term of the progression

$$1.4.7.10.13.16.19 \ldots\ldots,$$

we have $$l=1+49\times 3=148.$$

13. To find the 60th term of the progression

$$1.5.9.13.17.21.25 \ldots\ldots,$$

we have $$l=1+59\times 4=237.$$

180. If the progression were a decreasing one, we should have
$$l=a-(n-1)d.$$

Hence, to find the last term of a decreasing progression, we have the following rule: —

Multiply the common difference by the number of terms less one.

Subtract the product from the first term. The remainder will be the last term.

Exercises.

1. The first term of a decreasing progression is 60, the number of terms 20, and the common difference 3. What is the last term?

$$l = a - (n-1)d \text{ gives } l = 60 - (20-1)3 = 60 - 57 = 3.$$

2. The first term is 90, the common difference 4, and the number of terms 15. What is the last term? *Ans.* 34.

3. The first term is 100, the number of terms 40, and the common difference 2. What is the last term? *Ans.* 22.

4. The first term is 80, the number of terms 10, and the common difference 4. What is the last term? *Ans.* 44.

5. The first term is 600, the number of terms 100, and the common difference 5. What is the last term? *Ans.* 105.

6. The first term is 800, the number of terms 200, and the common difference 2. What is the last term? *Ans.* 402.

181. A progression by differences being given, it is proposed to prove that *the sum of any two terms, taken at equal distances from the two extremes, is equal to the sum of the two extremes;* that is, if we have the progression

$$2 . 4 . 6 . 8 . 10 . 12,$$

we wish to prove generally that

$$4 + 10, \text{ or } 6 + 8,$$

is equal to the sum of the two extremes, 2 and 12.

Let $a . b . c . e . f \ldots\ldots i . k . l$ be the proposed progression, and n the number of terms.

We will first observe that if x denotes a term which has

p terms before it, and y a term which has p terms after it, we have, from what has been said,

$$x = a + p \times d,$$

and $$y = l - p \times d;$$

whence, by addition, $x + y = a + l$,

which proves the proposition.

Referring to the previous example, if we suppose, in the first place, x to denote the second term 4, then y will denote the term 10, next to the last. If x denotes the third term 6, then y will denote 8, the third term from the last.

To apply this principle in finding the sum of the terms of a progression, write the terms as below, and then again in an inverse order; viz., —

$$a \,.\, b \,.\, c \,.\, d \,.\, e \,.\, f \ldots\ldots i \,.\, k \,.\, l.$$
$$l \,.\, k \,.\, i \ldots\ldots\ldots\ldots\ldots\ldots c \,.\, b \,.\, a.$$

Calling S the sum of the terms of the first progression, $2S$ will be the sum of the terms of both progressions, and we shall have

$$2S = (a+l) + (b+k) + (c+i) \ldots\ldots + (i+c) + (k+b) + (l+a).$$

Now, since all the parts, $a+l$, $b+k$, $c+i$ ....., are equal to each other, and their number equal to n,

$$2S = (a+l) \times n, \text{ or } S = \left(\frac{a+l}{2}\right) \times n.$$

Hence, for finding the sum of an arithmetical series, we have the following rule: —

Add the two extremes together, and take half their sum.

Multiply this half sum by the number of terms. The product will be the sum of the series.

Exercises.

1. The extremes are 2 and 16, and the number of terms 8. What is the sum of the series?

$$S=\left(\frac{a+l}{2}\right)\times n \text{ gives } S=\frac{2+16}{2}\times 8=72.$$

2. The extremes are 3 and 27, and the number of terms 12. What is the sum of the series? *Ans.* 180.

3. The extremes are 4 and 20, and the number of terms 10. What is the sum of the series? *Ans.* 120.

4. The extremes are 100 and 200, and the number of terms 80. What is the sum of the series? *Ans.* 12000.

5. The extremes are 500 and 60, and the number of terms 20. What is the sum of the series? *Ans.* 5600.

6. The extremes are 800 and 1200, and the number of terms 50. What is the sum of the series? *Ans.* 50000.

182. In arithmetical proportion there are five quantities to be considered, — first, the first term, a; second, the common difference, d; third, the number of terms, n; fourth, the last term, l; fifth, the sum, S.

The formulas

$$l=a+(n-1)d, \text{ and } S=\left(\frac{a+l}{2}\right)\times n,$$

contain five quantities, a, d, n, l, and S, and consequently give rise to the following general problem; viz., —

Any three of these five quantities being given, to determine the other two.

We already know the value of S in terms of a, n, and l.

From the formula

$$l = a + (n - 1)\,d$$

we find $$a = l - (n - 1)\,d; \text{ that is,}$$

The first term of an increasing arithmetical progression is equal to the last term, minus the product of the common difference by the number of terms less one.

From the same formula we also find

$$d = \frac{l - a}{n - 1}; \text{ that is,}$$

In any arithmetical progression, the common difference is equal to the last term, minus the first term, divided by the number of terms less one.

(1) The last term is 16, the first term 4, and the number of terms 5. What is the common difference?

The formula $$d = \frac{l - a}{n - 1}$$

gives $$d = \frac{16 - 4}{4} = 3.$$

(2) The last term is 22, the first term 4, and the number of terms 10. What is the common difference? *Ans.* 2.

183. The last principle affords a solution to the following question: —

To find a number m *of arithmetical means between two given numbers* a *and* b.

To resolve this question, it is first necessary to find the common difference. Now, we may regard a as the first term of an arithmetical progression, b as the last term, and the required means as intermediate terms. The number of terms of this progression will be expressed by $m + 2$.

Now, by substituting in the above formula, b for l, and $m+2$ for n, it becomes

$$d=\frac{b-a}{m+2-1}=\frac{b-a}{m+1};\text{ that is,}$$

The common difference of the required progression is obtained by dividing the difference between the given numbers, a *and* b, *by the required number of means plus one.*

Having obtained the common difference, d, form the second term of the progression, or the *first arithmetical mean*, by adding d to the first term a. The *second mean* is obtained by augmenting the first mean by d, etc.

(1) Find three arithmetical means between the extremes 2 and 18.

The formula $$d=\frac{b-a}{m+1}$$

gives $$d=\frac{18-2}{4}=4.$$

Hence the progression is $2 . 6 . 10 . 14 . 18$.

(2) Find twelve arithmetical means between 12 and 77.

The formula $$d=\frac{b-a}{m+1}$$

gives $$d=\frac{77-12}{13}=5.$$

Hence the progression is $12 . 17 . 22 . 27 \ldots\ldots 77$.

184. If the same number of arithmetical means are inserted between all the terms, taken two and two, these terms, and the arithmetical means united, will form one and the same progression.

Let $a . b . c . e . f \ldots\ldots$ be the proposed progression, and m the number of means to be inserted between a and b, b and c, c and e ....., etc.

From what has just been said, the common difference of each partial progression will be expressed by

$$\frac{b-a}{m+1},\ \frac{c-b}{m+1},\ \frac{e-c}{m+1},\ \ldots\ldots,$$

expressions which are equal to each other, since a, b, c, ... are in progression. Therefore the common difference is the same in each of the partial progressions; and, since the *last term* of the first forms the *first term* of the second, etc., we may conclude that all of these partial progressions form a single progression.

Exercises.

1. Find the sum of the first fifty terms of the progression $2 . 9 . 16 . 23 \ldots\ldots$

For the 50th term we have

$$l = 2 + 49 \times 7 = 345.$$

Hence $$S = (2 + 345) \times \frac{50}{2} = 347 \times 25 = 8675.$$

2. Find the 100th term of the series $2 . 9 . 16 . 23 \ldots\ldots$

Ans. 695.

3. Find the sum of 100 terms of the series $1 . 3 . 5 . 7 . 9 \ldots\ldots$

Ans. 10000.

4. The greatest term is 70, the common difference 3, and the number of terms 21. What is the least term, and the sum of the series? *Ans.* Least term, 10; sum of series, 840.

5. The first term is 4, the common difference 8, and the number of terms 8. What is the last term, and the sum of the series? *Ans.* Last term, 60; sum of series, 256.

6. The first term is 2, the last term 20, and the number of terms 10. What is the common difference? *Ans.* 2.

7. Insert four means between the two numbers 4 and 19. What is the series? *Ans.* $4 . 7 . 10 . 13 . 16 . 19.$

8. The first term of a decreasing arithmetical progression is 10, the common difference $\frac{1}{3}$, and the number of terms 21. Required the sum of the series. *Ans.* 140.

9. In a progression by differences, having given the common difference 6, the last term 185, and the sum of the terms 2945, find the first term and the number of terms.

Ans. 1st term, 5; number of terms, 31.

10. Find nine arithmetical means between each antecedent and consequent of the progression 2 . 5 . 8 . 11 . 14 .....

Ans. Common difference, or d, 0.3.

11. Find the number of men contained in a triangular battalion, the first rank containing 1 man, the second 2 men, the third 3, and so on to the nth, which contains n: in other words, find the expression for the sum of the natural numbers 1, 2, 3 ....., from 1 to n inclusively. *Ans.* $S=\frac{n(n+1)}{2}$.

12. Find the sum of the n first terms of the progression of uneven numbers 1 . 3 . 5 . 7 . 9 ..... *Ans.* $S=n^2$.

13. One hundred stones being placed on the ground in a straight line at the distance of 2 yards apart, how far will a person travel who shall bring them one by one to a basket placed at a distance of 2 yards from the first stone?

Ans. 11 miles, 840 yards.

Geometrical Proportion and Progression.

185. Ratio is the quotient arising from dividing one quantity by another quantity of the same kind, regarded as a standard. Thus, if the numbers 3 and 6 have the same unit, the ratio of 3 to 6 will be expressed by

$$\frac{6}{3}=2;$$

and in general, if A and B represent quantities of the same kind, the ratio of A to B will be expressed by

$$\frac{B}{A}.$$

186. The character $\propto$ indicates that one quantity varies as another. Thus,

$$A \propto B$$

is read, "A varies as B;" that is to say, the ratio of A to B, or $\frac{B}{A}$, is *constant*, though A and B themselves may change value.

If there are four numbers, 2, 4, 8, 16, having such values that the second divided by the first is equal to the fourth divided by the third, the numbers are said to form a proportion; and in general, if there are four quantities, A, B, C, and D, having such values that

$$\frac{B}{A}=\frac{D}{C},$$

then A is said to have the same ratio to B that C has to D, or the ratio of A to B is equal to the ratio of C to D. When four quantities have this relation to each other, compared together two and two, they are said to form a geometrical proportion.

To express that the ratio of A to B is equal to the ratio of C to D, we write the quantities thus:—

$$A : B :: C : D,$$

and read, "A is to B as C to D."

The quantities which are compared the one with the other are called **terms** of the proportion. The first and the last terms are called the **two extremes**; and the second and the third terms, the **two means.** Thus, A and D are the extremes; and B and C the means.

187. Of four terms of a proportion, the first and the third are called the **antecedents**; and the second and the fourth, the **consequents**; and the last is said to be a fourth proportional to the other three taken in order. Thus, in the last proportion, A and C are the antecedents, and B and D the consequents.

188. Three quantities are in proportion when the first has the same ratio to the second that the second has to the third; and then the middle term is said to be a mean proportional between the other quantities. For example,

$$3 : 6 :: 6 : 12;$$

and 6 is a mean proportional between 3 and 12.

189. Four quantities are said to be in proportion by **inversion**, or **inversely**, when the consequents are made the antecedents, and the antecedents the consequents.

Thus, if we have the proportion

$$3 : 6 :: 8 : 16,$$

the inverse proportion would be

$$6 : 3 :: 16 : 8.$$

190. Quantities are said to be in proportion by **alternation**, or **alternately**, when antecedent is compared with antecedent, and consequent with consequent.

Thus, if we have the proportion

$$3 : 6 :: 8 : 16,$$

the alternate proportion would be

$$3 : 8 :: 6 : 16.$$

191. Quantities are said to be in proportion by **composition** when the sum of the antecedent and consequent is compared with either antecedent or consequent.

Thus, if we have the proportion

$$2:4::8:16,$$

the proportion by composition would be

$$2+4:2::8+16:8;$$

and $$2+4:4::8+16:16.$$

192. Quantities are said to be in proportion by **division** when the difference of the antecedent and consequent is compared with either antecedent or consequent.

Thus, if we have the proportion

$$3:9::12:36,$$

the proportion by division will be

$$9-3:3::36-12:12$$

and $$9-3:9::36-12:36.$$

193. Equi-multiples of two or more quantities are the products which arise from multiplying the quantities by the same number.

Thus, if we have any two numbers, as 6 and 5, and multiply each of them by any number, as 9, the equi-multiples will be 54 and 45: for

$$6 \times 9 = 54, \text{ and } 5 \times 9 = 45.$$

Also $m \times A$ and $m \times B$ are equi-multiples of A and B, the common multiplier being m.

194. Two quantities, A and B, which may change their values, *are reciprocally or inversely proportional when one is*

proportional to unity divided by the other, and then their product remains constant.

We express this reciprocal or inverse relation thus:

$$A \propto \frac{1}{B},$$

in which A is said to be inversely proportional to B, or A varies as the reciprocal of B.

195. If we have the proportion

$$A : B :: C : D,$$

we have $$\frac{B}{A} = \frac{D}{C} \text{ (§ 186)};$$

and by clearing the equation of fractions we have

$$BC = AD; \text{ that is,}$$

Of four proportional quantities, the product of the two extremes is equal to the product of the two means.

This general principle is apparent in the proportion between the numbers

$$2 : 10 :: 12 : 60,$$

which gives $$2 \times 60 = 10 \times 12 = 120.$$

196. If four quantities, A, B, C, D, are so related to each other that

$$A \times D = B \times C,$$

we shall also have $$\frac{B}{A} = \frac{D}{C};$$

and hence $$A : B :: C : D; \text{ that is,}$$

If the product of two quantities is equal to the product of two other quantities, two of them may be made the extremes, and the other two the means, of a proportion.

Thus, if we have $\quad 2 \times 8 = 4 \times 4,$

we also have $\quad 2 : 4 : : 4 : 8.$

197. If we have three proportional quantities,

$$A : B : : B : C,$$

we have

$$\frac{B}{A} = \frac{C}{B}.$$

Hence $\quad B^2 = AC;$ that is,

If three quantities are proportional, the square of the middle term is equal to the product of the two extremes.

Thus, if we have the proportion

$$3 : 6 : : 6 : 12,$$

we shall also have

$$6 \times 6 = 6^2 = 3 \times 12 = 36.$$

198. If we have $\quad A : B : : C : D,$

and consequently

$$\frac{B}{A} = \frac{D}{C},$$

multiply both members of the last equation by $\frac{C}{B}$, and we then obtain

$$\frac{C}{A} = \frac{D}{B};$$

and hence $\quad A : C : : B : D;$ that is,

If four quantities are proportional, they will be in proportion by alternation.

Let us take as an example

$$10 : 15 : : 20 : 30.$$

We shall have, by alternating the terms,

$$10 : 20 : : 15 : 30.$$

199. If we have

$$A : B :: C : D, \text{ and } A : B :: E : F,$$

we shall also have

$$\frac{B}{A} = \frac{D}{C}, \text{ and } \frac{B}{A} = \frac{F}{E}.$$

Hence $\frac{D}{C} = \frac{F}{E}$, and $C : D :: E : F$; that is,

If there are two sets of proportions having an antecedent and consequent of the one equal to an antecedent and consequent of the other, the remaining terms will be proportional.

If we have the two proportions

$$2 : 6 :: 8 : 24, \text{ and } 2 : 6 :: 10 : 30,$$

we shall also have

$$8 : 24 :: 10 : 30.$$

200. If we have

$$A : B :: C : D, \text{ and consequently } \frac{B}{A} = \frac{D}{C},$$

we have, by dividing 1 by each member of the equation,

$\frac{A}{B} = \frac{C}{D}$, and consequently $B : A :: D : C$; that is,

Four proportional quantities will be in proportion when taken inversely.

To give an example in numbers, take the proportion

$$7 : 14 :: 8 : 16.$$

Then the inverse proportion will be

$$14 : 7 :: 16 : 8,$$

in which the ratio is one half.

201. The proportion

$$A : B :: C : D \text{ gives } A \times D = B \times C.$$

To each member of the last equation add $B \times D$. We shall then have

$$(A + B) \times D = (C + D) \times B;$$

and by separating the factors we obtain

$$A + B : B :: C + D : D.$$

If, instead of adding, we subtract $B \times D$ from both members, we have

$$(A - B) \times D = (C - D) \times B,$$

which gives $\quad A - B : B :: C - D : D;$ that is,

If four quantities are proportional, they will be in proportion by composition or division.

Thus, if we have the proportion

$$9 : 27 :: 16 : 48,$$

we shall have, by composition,

$$9 + 27 : 27 :: 16 + 48 : 48;$$

that is, $\quad 36 : 27 :: 64 : 48,$

in which the ratio is three fourths.

The same proportion gives us, by division,

$$27 - 9 : 27 :: 48 - 16 : 48;$$

that is, $\quad 18 : 27 :: 32 : 48,$

in which the ratio is one and one half.

202. If we have

$$\frac{B}{A}=\frac{D}{C},$$

and multiply the numerator and denominator of the first member by any number, m, we obtain

$$\frac{mB}{mA}=\frac{D}{C},$$

and $$mA : mB :: C : D;$$ that is,

Equi-multiples of two quantities have the same ratio as the quantities themselves.

For example: if we have the proportion

$$5 : 10 :: 12 : 24,$$

and multiply the first antecedent and consequent by 6, we have

$$30 : 60 :: 12 : 24,$$

in which the ratio is still 2.

203. The proportions

$$A : B :: C : D$$

and $$A : B :: E : F$$

give $$A \times D = B \times C,$$

and $$A \times F = B \times E.$$

Adding and subtracting these equations, we obtain

$$A(D \pm F) = B(C \pm E),$$

or $$A : B :: C \pm E : D \pm F;$$ that is,

If C *and* D, *the antecedent and consequent, be augmented or diminished by quantities* E *and* F, *which have the same ratio as* C *to* D, *the resulting quantities will also have the same ratio.*

Let us take as an example the proportion $9 : 18 :: 20 : 40$, in which the ratio is 2.

If we augment the antecedent and consequent by the numbers 15 and 30, which have the same ratio, we shall have $9 + 15 : 18 + 30 :: 20 : 40$; that is, $24 : 48 :: 20 : 40$, in which the ratio is still 2.

If we diminish the second antecedent and consequent by these numbers respectively, we have $9 : 18 :: 20 - 15 : 40 - 30$; that is, $9 : 18 :: 5 : 10$, in which the ratio is still 2.

204. If we have several proportions,

$A : B :: C : D$, which gives $A \times D = B \times C$,

$A : B :: E : F$, which gives $A \times F = B \times E$,

$A : B :: G : H$, which gives $A \times H = B \times G$, etc.,

we shall have, by addition,

$$A(D + F + H) = B(C + E + G);$$

and by separating the factors,

$$A : B :: C + E + G : D + F + H; \text{ that is,}$$

In any number of proportions having the same ratio, any antecedent will be to its consequent as the sum of the antecedents to the sum of the consequents.

Let us take, for example,

$$2 : 4 :: 6 : 12, \text{ and } 1 : 2 :: 3 : 6, \text{ etc.}$$

Then $$2 : 4 :: 6 + 3 : 12 + 6;$$

that is, $$2 : 4 :: 9 : 18,$$

in which the ratio is still 2.

205. If we have four proportional quantities,

$$A : B :: C : D, \text{ we have } \frac{B}{A} = \frac{D}{C};$$

and raising both members to any power whose exponent is n, or extracting any root whose index is n, we have

$$\frac{B^n}{A^n} = \frac{D^n}{C^n}, \text{ or } \frac{B^{\frac{1}{n}}}{A^{\frac{1}{n}}} = \frac{D^{\frac{1}{n}}}{C^{\frac{1}{n}}};$$

and consequently

$$A^n : B^n :: C^n : D^n, \text{ or } A^{\frac{1}{n}} : B^{\frac{1}{n}} :: C^{\frac{1}{n}} : D^{\frac{1}{n}}; \text{ that is,}$$

If four quantities are proportional, their like powers or roots will be proportional.

If we have, for example,

$$2 : 4 :: 3 : 6,$$

we shall have $\quad 2^2 : 4^2 :: 3^2 : 6^2;$

that is, $\quad 4 : 16 :: 9 : 36,$

in which the terms are proportional, the ratio being 4.

206. Let there be two sets of proportions, —

$$A : B :: C : D, \text{ which gives } \frac{B}{A} = \frac{D}{C};$$

$$E : F :: G : H, \text{ which gives } \frac{F}{E} = \frac{H}{G}.$$

Multiplying them together member by member, we have,

$$\frac{B \times F}{A \times E} = \frac{D \times H}{C \times G},$$

$$A \times E : B \times F :: C \times G : D \times H; \text{ that is,}$$

In two sets of proportional quantities, the products of the corresponding terms are proportional.

Thus, if we have the two proportions

$$8 : 16 :: 10 : 20,$$

and

$$3 : 4 :: 6 : 8,$$

we shall have

$$24 : 64 :: 60 : 160.$$

207. We have thus far considered only the case in which the ratio of the first term to the second is the same as that of the third to the fourth.

If we have the further condition that the ratio of the second term to the third shall also be the same as that of the first to the second or of the third to the fourth, we shall have a series of numbers each one of which, divided by the preceding one, will give the same ratio. Hence, if any term be multiplied by this quotient, the product will be the succeeding term. A series of numbers so formed is called a **geometrical progression.** Hence

A **geometrical progression,** or **progression by quotients,** is a series of terms each of which is equal to the preceding term multiplied by a *constant number*, which number is called the *ratio* of the progression. Thus,

$$1 : 3 : 9 : 27 : 81 : 243, \text{ etc.},$$

is a geometrical progression in which the ratio is 3. It is written by merely placing two dots between the terms.

Also

$$64 : 32 : 16 : 8 : 4 : 2 : 1$$

is a geometrical progression in which the ratio is *one half.*

In the first progression each term is contained three times in the one that follows, and hence the ratio is 3. In the second, each term is contained one half times in the one which follows, and hence the ratio is one half.

The first is called an **increasing** progression; and the second, a **decreasing** progression.

Let a, b, c, d, e, f ..... be numbers in a progression by quotients. They are written thus: —

$$a : b : c : d : e : f : g \ldots\ldots,$$

and it is enunciated in the same manner as a progression by differences. It is necessary, however, to make the distinction that one is a series formed by equal differences, and the other a series formed by equal quotients or ratios. It should be remarked that each term is at the same time an antecedent and a consequent, except the first, which is only an antecedent, and the last, which is only a consequent.

208. Let r denote the ratio of the progression

$$a : b : c : d \ldots\ldots,$$

r being >1 when the progression is *increasing*, and $r<1$ when it is *decreasing*. Then, since

$$\frac{b}{a} = r,\ \frac{c}{b} = r,\ \frac{d}{c} = r,\ \frac{e}{d} = r,\ \text{etc.},$$

we have

$$b = ar,\ c = br = ar^2,\ d = cr = ar^3,$$

$$e = dr = ar^4,\ f = er = ar^5, \ldots\ldots;$$

that is, the second term is equal to ar, the third to ar^2, the fourth to ar^3, the fifth to ar^4, etc.; and in general the nth term, that is, one which has $n-1$ terms before it, is expressed by ar^{n-1}.

Let l be this term. We then have the formula

$$l = ar^{n-1},$$

by means of which we can obtain any term without being obliged to find all the terms which precede it. Hence, to find the last term of a progression, we have the following rule.

Raise the ratio to a power whose exponent is one less than the number of terms.

Multiply the power thus found by the first term. The product will be the required term.

Exercises.

1. Find the 5th term of the progression $2:4:8:16$ ....., in which the first term is 2, and the common ratio 2.

$$5\text{th term} = 2 \times 2^4 = 2 \times 16 = 32, \textit{ Ans.}$$

2. Find the 8th term of the progression $2:6:18:54$ .....

$$8\text{th term} = 2 \times 3^7 = 2 \times 2187 = 4374, \textit{ Ans.}$$

3. Find the 6th term of the progression $2:8:32:128$ .....

$$6\text{th term} = 2 \times 4^5 = 2 \times 1024 = 2048, \textit{ Ans.}$$

4. Find the 7th term of the progression $3:9:27:81$ .....

$$7\text{th term} = 3 \times 3^6 = 3 \times 729 = 2187, \textit{ Ans.}$$

5. Find the 6th term of the progression $4:12:36:108$ .....

$$6\text{th term} = 4 \times 3^5 = 4 \times 243 = 972, \textit{ Ans.}$$

6. A person agreed to pay his servant one cent for the first day, two for the second, and four for the third, doubling every day for ten days. How much did he receive on the tenth day? *Ans.* \$5.12.

7. What is the 8th term of the progression

$$9:36:144:576 \text{ .....?}$$

$$8\text{th term} = 9 \times 4^7 = 9 \times 16384 = 147456, \textit{ Ans.}$$

8. Find the 12th term of the progression

$$64:16:4:1:\frac{1}{4} \text{ .....}$$

$$12\text{th term} = 64\left(\frac{1}{4}\right)^{1} = \frac{4^3}{4^{11}} = \frac{1}{4^8} = \frac{1}{65536}, \textit{ Ans.}$$

209. We will now proceed to determine the sum of n terms of a progression,

$$a : b : c : d : e : f : \ldots\ldots : i : k : l,$$

l denoting the nth term.

We have the equations (§ 208)

$$b = ar,\ c = br,\ d = cr,\ e = dr, \ldots\ldots k = ir,\ l = kr;$$

and by adding them all together, member to member, we deduce

$$\overset{\textit{Sum of 1st Members.}}{b + c + d + e + \cdots\cdots + k + l} = \overset{\textit{Sum of 2d Members.}}{(a + b + c + d + \cdots\cdots + i + k)r},$$

in which we see that the first member contains all the terms but a, and the polynomial within the parenthesis in the second member contains all the terms but l. Hence, if we call the sum of the terms S, we have

$$S - a = (S - l)r = Sr - lr,$$

$$\therefore\ Sr - S = lr - a;$$

whence $$S = \frac{lr - a}{r - 1}.$$

Therefore, to obtain the sum of all the terms, or sum of the series of a geometrical progression, we have the rule: —

Multiply the last term by the ratio.

Subtract the first term from the product.

Divide the remainder by the ratio diminished by 1, *and the quotient will be the sum of the series.*

Exercises.

1. Find the sum of eight terms of the progression

$$2 : 6 : 18 : 54 : 162 \ldots\ldots 2 \times 3^7 = 4374.$$

$$S = \frac{lr - a}{r - 1} = \frac{13122 - 2}{2} = 6560, \textit{ Ans.}$$

2. Find the sum of the progression $2:4:8:16:32$.

$$S=\frac{lr-a}{r-1}=\frac{64-2}{1}=62, \textit{ Ans.}$$

3. Find the sum of ten terms of the progression

$$2:6:18:54:162\ldots\ldots 2\times 3^9=39366.$$

Ans. 59048.

4. What debt may be discharged in a year, or twelve months, by paying \$1 the first month, \$2 the second month, \$4 the third month, and so on, each succeeding payment being double the last; and what will be the last payment?

Ans. Debt, \$4095; last payment, \$2048.

5. A daughter was married on New-Year's Day. Her father gave her 1s., with an agreement to double it on the first of the next month, and at the beginning of each succeeding month to double what she had received the previous month. How much had she received at the end of the year?

Ans. £204 15s.

6. A man bought 10 bushels of wheat on the condition that he should pay 1 cent for the first bushel, 3 for the second, 9 for the third, and so on to the last. What did he pay for the last bushel, and what for the ten bushels?

Ans. Last bushel, \$196.83; total cost, \$295.24.

7. A man plants 4 bushels of barley, which at the first harvest produced 32 bushels; these he also plants, which, in like manner, produce eightfold; he again plants all his crop, and again gets eightfold; and so on for 16 years. What is his last crop, and what the sum of the series?

Ans. Last, 140,737,488,355,328 bushels;
sum, 160,842,843,834,660.

210. When the progression is decreasing, we have $r < 1$, and $l < a$. The above formula,

$$S = \frac{lr - a}{r - 1},$$

for the sum, is then written under the form

$$S = \frac{a - lr}{1 - r},$$

in order that the two terms of the fraction may be positive.

Exercises.

1. Find the sum of the terms of the progression

$$32 : 16 : 8 : 4 : 2.$$

$$S = \frac{a - lr}{1 - r} = \frac{32 - 2 \times \frac{1}{2}}{\frac{1}{2}} = \frac{31}{\frac{1}{2}} = 62, \textit{ Ans.}$$

2. Find the sum of the first twelve terms of the progression

$$64 : 16 : 4 : 1 : \frac{1}{4} : \ldots\ldots : 64\left(\frac{1}{4}\right)^{11}, \text{ or } \frac{1}{65536}.$$

$$S = \frac{a - lr}{1 - r} = \frac{64 - \frac{1}{65536} \times \frac{1}{4}}{\frac{3}{4}} = \frac{256 - \frac{1}{65536}}{3} = 85 + \frac{65535}{196608}.$$

211. Note. — We perceive that the principal difficulty consists in obtaining the numerical value of the last term, — a tedious operation, even when the number of terms is not very great.

3. Find the sum of six terms of the progression

$$512 : 128 : 32 \ldots\ldots$$

Ans. $682\frac{1}{2}$.

4. Find the sum of seven terms of the progression

$$2187 : 729 : 243 \ldots\ldots$$

Ans. 3279.

5. Find the sum of six terms of the progression

$$972 : 324 : 108 \ldots\ldots$$

Ans. 1456.

6. Find the sum of eight terms of the progression

$$147456 : 36864 : 9216 \ldots\ldots$$

Ans. 196605.

212. Let there be the decreasing progression

$$a : b : c : d : e : f : \ldots\ldots,$$

containing an indefinite number of terms. In the formula

$$S = \frac{a - lr}{1 - r}$$

substitute for l its value ar^{n-1} (§ 208), and we have

$$S = \frac{a - ar^n}{1 - r},$$

which expresses the sum of n terms of the progression. This may be put under the form

$$S = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

Now, since the progression is decreasing, r is a proper fraction; and r^n is also a fraction, which diminishes as n increases. Therefore the greater the number of terms we take, the more will $\frac{a}{1-r} \times r^n$ diminish, and consequently the more will the entire sum of all the terms approximate to an equality with the first part of S; that is, to $\frac{a}{1-r}$. Finally, when n is taken greater than any given number, or $n =$ infinity, then

$\frac{a}{1-r} \times r^n$ will be less than any given number, or will differ so little from 0 that its value may be disregarded without making any appreciable difference in the result; and the expression $\frac{a}{1-r}$ will then represent the true value of the sum of all the terms of the series. Whence we may conclude that the expression for *the sum of the terms of a decreasing progression, in which the number of terms is infinite, is*

$$S = \frac{a}{1-r};$$

that is, *equal to the first term, divided by* 1 *minus the ratio.*

That is, properly speaking, the *limit* to which the *partial sums* approach, as we take a greater number of terms in the progression. The difference between these sums and $\frac{a}{1-r}$ may be made as small as we please, but will only become *inappreciable* when the number of terms is infinite.

Exercises.

1. Find the sum of $1 : \frac{1}{3} : \frac{1}{9} : \frac{1}{27} : \frac{1}{81}$ to infinity.

We have, for the expression of the sum of the terms,

$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} = 1\tfrac{1}{2}, \textit{ Ans.}$$

The error committed by taking this expression for the value of the sum of the n first terms is expressed by

$$\frac{a}{1-r} \times r^n = \frac{3}{2}\left(\frac{1}{3}\right)^n.$$

First take $n = 5$, it becomes

$$\frac{3}{2}\left(\frac{1}{3}\right)^5 = \frac{1}{2 \cdot 3^4} = \frac{1}{162}.$$

When $n = 6$, we find

$$\frac{3}{2}\left(\frac{1}{3}\right)^6 = \frac{1}{162} \times \frac{1}{3} = \frac{1}{486}.$$

The *error committed* by taking $\frac{3}{2}$ for the sum of a number of terms is evidently less in proportion as the number of terms is greater.

2. Again take the progression

$$1 : \frac{1}{2} : \frac{1}{4} : \frac{1}{8} : \frac{1}{16} : \frac{1}{32} : \text{etc.} \ldots\ldots$$

$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2, \ \textit{Ans.}$$

3. What is the sum of the progression

$$1, \ \frac{1}{10}, \ \frac{1}{100}, \ \frac{1}{1000}, \ \frac{1}{10000}, \text{ etc., to infinity?}$$

$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{10}} = 1\tfrac{1}{9}, \ \textit{Ans.}$$

213. In the several questions of geometrical progression there are five numbers to be considered, — first, the first term, a; second, the ratio, r; third, the number of terms, n; fourth, the last term, l; fifth, the sum of the terms, S.

214. We shall terminate this subject by solving this problem : —

Find a mean proportional between any two numbers, as m and n.

Denote the required mean by x. We shall then have (§ 197)

$$x^2 = m \times n,$$

and hence $\qquad x = \sqrt{m \times n}$; that is,

Multiply the two numbers together, and extract the square root of the product.

Exercises.

1. What is the geometrical mean between the numbers 2 and 8?

$$\text{Mean} = \sqrt{8 \times 2} = \sqrt{16} = 4, \quad \textit{Ans.}$$

2. What is the mean between 4 and 16? *Ans.* 8.

3. What is the mean between 3 and 27? *Ans.* 9.

4. What is the mean between 2 and 72? *Ans.* 12.

5. What is the mean between 4 and 64? *Ans.* 16.

CHAPTER X.

LOGARITHMS.

215. The nature and properties of the logarithms in common use will be readily understood by considering attentively the different powers of the number 10. They are

$$10^0 = 1,$$
$$10^1 = 10,$$
$$10^2 = 100,$$
$$10^3 = 1000,$$
$$10^4 = 10000,$$
$$10^5 = 100000, \text{ etc.}$$

It is plain that the exponents 0, 1, 2, 3, 4, 5, etc., form an arithmetical progression of which the common difference is 1; and that the corresponding numbers 1, 10, 100, 1000, 10000, 100000, etc., form a geometrical progression of which the common ratio is 10. The number 10 is called the **base** of the system of logarithms; and the exponents 0, 1, 2, 3, 4, 5, etc., are the **logarithms** of the numbers which are produced by raising 10 to the powers denoted by those exponents.

216. If we denote the logarithm of any number by m, then the number itself will be the mth power of 10; that is, if we represent the corresponding number by M,

$$10^m = M.$$

Thus, if we make $m = 0$, M will be equal to 1; if $m = 1$, M will be equal to 10; etc. Hence

The logarithm of a number is the exponent of the power to which it is necessary to raise the base of the system in order to produce the number.

217. If, as before, 10 denotes the base of the system of logarithms, m any exponent, and M the corresponding number, we shall then have

$$10^m = M \qquad (1)$$

in which m is the logarithm of M.

If we take a second exponent, n, and let N denote the corresponding number, we shall have

$$10^n = N \qquad (2)$$

in which n is the logarithm of N.

If, now, we multiply the first of these equations by the second, member by member, we have

$$10^m \times 10^n = 10^{m+n} = M \times N.$$

But, since 10 is the base of the system, $m+n$ is the logarithm of $M \times N$. Hence

The sum of the logarithms of any two numbers is equal to the logarithm of their product.

Therefore the addition of logarithms corresponds to the multiplication of their numbers.

218. If we divide Equation (1) by Equation (2), member by member, we have

$$\frac{10^m}{10^n} = 10^{m-n} = \frac{M}{N}.$$

But, since 10 is the base of the system, $m-n$ is the logarithm of $\frac{M}{N}$. Hence

If one number be divided by another, the logarithm of the quotient will be equal to the logarithm of the dividend, diminished by that of the divisor.

Therefore the subtraction of logarithms corresponds to the division of their numbers.

219. Let us examine further the equations

$$10^0 = 1,$$
$$10^1 = 10,$$
$$10^2 = 100,$$
$$10^3 = 1000, \text{ etc.}$$

It is plain that the logarithm of 1 is 0, and that the logarithm of any number between 1 and 10 is greater than 0 and less than 1. The logarithm is generally expressed by decimal fractions. Thus,

$$\log 2 = 0.301030.$$

The logarithm of any number greater than 10 and less than 100 is greater than 1 and less than 2, and is expressed by 1 and a decimal fraction. Thus,

$$\log 50 = 1.698970.$$

The part of the logarithm which stands at the *left* of the decimal point is called the **characteristic** of the logarithm, and the part that stands at the *right* of the decimal point is called the **mantissa** of the logarithm. In a whole number, the characteristic is always *one less than the number of places of figures in the number whose logarithm is taken.*

Thus, in the first case, for numbers between 1 and 10 there is but one place of figures, and the characteristic is 0; for numbers between 10 and 100 there are two places of figures, and the characteristic is 1; and similarly for other numbers.

If we form the powers of 10 denoted by negative expo-

nents (see § 50), we have, from the definition of a logarithm, the following table: —

$$(10)^{-1} = \frac{1}{10} = .1; \quad \therefore \log .1 = -1.$$

$$(10)^{-2} = \frac{1}{100} = .01; \quad \therefore \log .01 = -2.$$

$$(10)^{-3} = \frac{1}{1000} = .001; \quad \therefore \log .001 = -3.$$

And so on.

Here we see that the logarithm of every number between 1 and .1 is found between 0 and -1; that is, it is equal to -1, *plus* a part less than 1. The logarithm of every number between .1 and .01 is between -1 and -2; that is, it is equal to -2, *plus* a part. The logarithm of every number between .01 and .001 is between -2 and -3, or it is equal to -3, *plus* a part. And so on.

In the first case the characteristic is -1; in the second, -2; in the third, -3; and so on. That is,

The characteristic of the logarithm of a decimal is negative, and numerically one greater than the number of ciphers that immediately follow the decimal point.

In a mixed number, the characteristic of its logarithm will evidently be *one less than the number of places of figures* in its *entire* part.

The decimal part of a logarithm is always regarded as positive; and, to indicate that the negative sign extends only to the characteristic, it is generally written over it. Thus,

$\log 0.012 = \bar{2}.079181$, equivalent to -2 prefixed to $+.079181$.

220. A table of logarithms is a table containing a set of numbers and their logarithms, so arranged that we may, by

its aid, find the logarithm corresponding to any number of the set or the number corresponding to any of the logarithms.

A table showing the logarithms of all whole numbers from 1 to 100 is annexed. The numbers are written in the column designated by the letter "N," and the logarithms in the column designated by "Log."

Table.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	26	1.414973	51	1.707570	76	1.880814
2	0.301030	27	1.431364	52	1.716003	77	1.886491
3	0.477121	28	1.447158	53	1.724276	78	1.892095
4	0.602060	29	1.462398	54	1.732394	79	1.897627
5	0.698970	30	1.477121	55	1.740363	80	1.903090
6	0.778151	31	1.491362	56	1.748188	81	1.908485
7	0.845098	32	1.505150	57	1.755875	82	1.913814
8	0.903090	33	1.518514	58	1.763428	83	1.919078
9	0.954243	34	1.531479	59	1.770852	84	1.924279
10	1.000000	35	1.544068	60	1.778151	85	1.929419
11	1.041393	36	1.556303	61	1.785330	86	1.934498
12	1.079181	37	1.568202	62	1.792392	87	1.939519
13	1.113943	38	1.579784	63	1.799341	88	1.944483
14	1.146128	39	1.591065	64	1.806180	89	1.949390
15	1.176091	40	1.602060	65	1.812913	90	1.954243
16	1.204120	41	1.621784	66	1.819544	91	1.959041
17	1.230449	42	1.623249	67	1.826075	92	1.963788
18	1.255273	43	1.633468	68	1.832509	93	1.968483
19	1.278754	44	1.643453	69	1.838849	94	1.973128
20	1.301030	45	1.653213	70	1.845098	95	1.977724
21	1.322219	46	1.662758	71	1.851258	96	1.982271
22	1.342423	47	1.672098	72	1.857333	97	1.986772
23	1.361728	48	1.681241	73	1.863323	98	1.991226
24	1.380211	49	1.690196	74	1.869232	99	1.995635
25	1.397940	50	1.698970	75	1.875061	100	2.000000

Exercises.

1. Let it be required to multiply 8 by 9 by means of logarithms.

NOTE. — We have seen (§ 216) that the sum of the logarithms is equal to the logarithm of the product: therefore find the logarithm of 8 from the table, which is 0.903090; and then the logarithm of 9, which is 0.954243; and their sum, which is 1.857333, will be the logarithm of the product. In searching along in the table, we find that 72 stands opposite this logarithm: hence 72 is the product of 8 by 9.

2. What is the product of 7 by 12?

Logarithm of 7	0.845098
Logarithm of 12	1.079181
Logarithm of their product	1.924279

Corresponding number, 84.

3. What is the product of 9 by 11?

Logarithm of 9	0.954243
Logarithm of 11	1.041393
Logarithm of their product	1.995636

Corresponding number, 99.

4. Let it be required to divide 84 by 3.

NOTE. — We have seen (§ 218) that the subtraction of logarithms corresponds to the division of their numbers: hence if we find the logarithm of 84, and then subtract from it the logarithm of 3, the remainder will be the logarithm of the quotient.

Logarithm of 84	1.924279
Logarithm of 3	0.477121
Difference	1.447158

Corresponding number, 28.

5. Divide 42 by 7.

Logarithm of 42	1.623249
Logarithm of 7	0.845098
Logarithm of the quotient	0.778151

Corresponding number, 6.

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